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Composition of dyadic paraproducts [☆]

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We obtain necessary and sufficient conditions to characterize the boundedness of the composition of dyadic paraproduct operators.

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Contents

1. Introduction

Recall that a Toeplitz operator on the Hardy space of analytic functions $H^2(\mathbb{D})$ is defined by

$$T_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D}) \text{ where } T_\varphi f = \mathbb{P}_{H^2} (\varphi f) .$$

It is well known that this operator is bounded if and only if $\varphi \in L^\infty(\mathbb{T})$. Equivalently, the Toeplitz operator T_φ is bounded if and only if $\sup_{\lambda\in\mathbb{D}} \|T_\varphi k_\lambda\|_{H^2} < \infty$ where $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$ is the reproducing kernel for $H^2(\mathbb{D})$. An infamous conjecture of Sarason, [8], states that the composition of two (potentially unbounded) Toeplitz operators is bounded, i.e. $T_\varphi T_{\overline{\psi}}$ is a bounded operator, if and only if a certain relatively simple testing condition on the symbols φ and ψ holds, see [10]. However, even though this conjecture seems quite reasonable, a beautiful counterexample was constructed by F. Nazarov in [2] disproving this simple testing condition.

In this paper we are interested in a discrete dyadic analogue of the Sarason conjecture. This discrete problem is already very challenging and captures much of the difficulty associated with Sarason's original conjecture but is more amenable to study because of the dyadic nature of the problem. In particular, we are concerned with dyadic Haar paraproducts, and obtaining necessary and sufficient conditions for the boundedness of the composition of two such paraproducts. The conditions characterizing the boundedness will be much more general than just those characterizing boundedness for each individual paraproduct – just as the condition $\|bd\|_\infty < \infty$ that characterizes boundedness of the composition $M_b \circ M_d$ of pointwise multipliers is much more general than the conditions $\|b\|_\infty < \infty$ and $\|d\|_\infty < \infty$ that characterize individual boundedness of the pointwise multipliers.

Let $\mathcal{D}$ denote the usual dyadic grid of intervals on the real line. We consider sequences $b = \{b_I\}_{I\in\mathcal{D}}$ of complex numbers on $\mathcal{D}$, which we often refer to as *symbols*. Define the Haar function h_I^0 and averaging function h_I^1 by

$$h_I^0 \equiv h_I \equiv \frac{1}{\sqrt{|I|}} \left(-\mathbf{1}_{I_-} + \mathbf{1}_{I_+}\right) \text{ and } h_I^1 \equiv \frac{1}{|I|}\mathbf{1}_I \ , \qquad I \in \mathcal{D}.$$

The operators considered in this paper are the following dyadic paraproducts.