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Second main theorem in the tropical projective space

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ABSTRACT

Tropical Nevanlinna theory, introduced by Halburd and Southall as a tool to analyze integrability of ultra-discrete equations, studies the growth and complexity of continuous piecewise linear real functions. The purpose of this paper is to extend tropical Nevanlinna theory to n-dimensional tropical projective spaces by introducing a natural characteristic function for tropical holomorphic curves, and by proving a tropical analogue of Cartan's second main theorem. It is also shown that in the 1-dimensional case this result implies a known tropical second main theorem due to Laine and Tohge.

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1. Introduction

Tropical Nevanlinna theory of continuous piecewise linear real functions, or of *tropical meromorphic functions*, was recently introduced by Halburd and Southall [10]. They defined tropical versions of the Nevanlinna functions, and showed that they share many of the properties of their classical counterparts [11,5], including Jensen's formula and an analogue of the first main theorem. Halburd and Southall applied tropical Nevanlinna theory to measure complexity of tropical meromorphic functions satisfying ultra-discrete equations. They suggested, analogously to the case of difference equations in the complex plane [1,8], that existence of sufficiently many finite-order tropical meromorphic solutions of an ultra-discrete equation is a necessary condition for the equation in question to be of Painlevé type. Laine and Yang [14] have laid the groundwork for the systematic study of value distribution of tropical meromorphic solutions of ultra-discrete equations by proving a number of general results applicable to large classes of ultra-discrete equations. These include a generalized ultra-discrete version of Clunie's lemma, and an analogue of Mohon'ko's lemma on value distribution of meromorphic solutions of differential equations. A study of general fundamental properties of tropical meromorphic functions has been performed by Tsai in [20]. Tsai mainly discusses the family of piecewise linear functions defined on the extended real line $\mathbb{R} \cup \{-\infty\}$, and he calls tropical meromorphic functions defined on $\mathbb{R}$ by the name $\mathbb{R}$-*tropical meromorphic*.

Laine and Tohge generalized tropical Nevanlinna theory to include piecewise linear functions with arbitrary real slopes, and proved a tropical version of the second main theorem for tropical meromorphic functions under a growth condition that is less restrictive than demanding finite order [13]. Their results imply that behaviour of tropical meromorphic functions is in certain respects fundamentally different from their classical counterparts in the sense of value distribution. On one hand, the tropical second main theorem due to Laine and Tohge implies that under a natural non-degeneracy condition tropical meromorphic functions of finite order have no deficient values. On the other hand, a meaningful ramification term for the second main theorem in the tropical setting is yet to be discovered.

The purpose of this study is to extend tropical Nevanlinna theory to tropical holomorphic curves in a finite dimensional tropical projective space. We introduce a tropical analogue of the Cartan characteristic function for tropical holomorphic curves, and show that it reduces to the Nevanlinna characteristic due to Halburd and Southall in the one-dimensional case. As a central result of the tropical Nevanlinna-Cartan theory, we introduce a tropical analogue of Cartan's second main theorem and show that it generalizes the second main theorem by Laine and Tohge [13]. In fact, our results imply a stronger version of the tropical second main theorem by Laine and Tohge in the sense that one of the conditions in their theorem can be deleted by using our tropical analogue of Cartan's second main theorem. This result, which is Theorem 6.3 below, implies also a second main theorem containing a tropical analogue of the ramification term expressible in terms of a tropical Casoratian.