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# Random countable iterated function systems with overlaps and applications

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A B S T R A C T

We study invariant measures for random countable (finite or infinite) conformal iterated function systems (IFS) with arbitrary overlaps. We do not assume any type of separation condition. We prove, under a mild assumption of finite entropy, the dimensional exactness of the projections of invariant measures from the shift space, and we give a formula for their dimension, in the context of random infinite conformal iterated function systems with overlaps. There exist many differences between our case and the finite deterministic case studied in [7], and we introduce new methods specific to the infinite and random case. We apply our results towards a problem related to a conjecture of Lyons about random continued fractions ([11]), and show that for all parameters $\lambda > 0$, the invariant measure $\nu_\lambda$ is exact dimensional; and in addition, we give estimates for the pointwise and Hausdorff dimension of $\nu_\lambda$, for $\lambda$ in a certain interval. The finite IFS determining these continued fractions is not hyperbolic, but we can associate to it a random infinite IFS of contractions which have overlaps. We study then also other large classes of random countable iterated function systems with overlaps, namely: a) several types of

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random iterated function systems related to Kahane–Salem sets; and b) randomized infinite IFS in the plane which have uniformly bounded number of disc overlaps. For all the above classes, we prove dimensional exactness, and we find lower and upper estimates for the pointwise (and Hausdorff, packing) dimensions of the projection measures.


Infinite IFS in the plane with disc overlaps
Families of conditional measures

## 1. Introduction

Let $(X, \rho)$ be a metric space. A finite Borel measure $\mu$ on $X$ is called *exact dimensional* if

$$d_\mu(x) := \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{1.1}$$

exists for $\mu$-a.e. $x \in X$ and is equal to a common value denoted by $d_\mu$. Exact dimensionality of the measure $\mu$ has profound geometric consequences (for e.g. [6,7,13,22,25]).

The question of which measures are exact dimensional attracted the attention at least since the seminal paper of L.S. Young [34], where it was proved a formula for the Hausdorff dimension of a hyperbolic measure invariant under a surface diffeomorphism, formula involving the Lyapunov exponents of the measure. As a consequence of that proof, she established what (now) is called the dimensional exactness of such measures. The topic of dimensional exactness was then pursued by the breakthrough result of Barreira, Pesin, and Schmeling who proved in [1] the Eckmann–Ruelle conjecture asserting that any hyperbolic measure invariant under smooth diffeomorphisms is exact dimensional ([4]). Dimensional exactness, without using these words, was also established in the book [14] for all projected invariant measures with finite entropy, in the setting of conformal iterated function systems with countable alphabet which satisfy the Open Set Condition (OSC); in particular for all projected invariant measures if the alphabet is finite and we have OSC. The next difficult task was the case of a conformal iterated function system with *overlaps*, i.e. without assuming the Open Set Condition. For the case of iterated function systems with finite alphabet and having overlaps, this was done by Feng and Hu in [7]. Overlaps in iterated function systems (IFS) are challenging. Our goal in the present paper is to extend the above mentioned paper of Feng and Hu, in *two directions*. Firstly, by allowing the alphabet of the system to be *countable infinite*; and secondly, to consider *random* iterated function systems rather than deterministic IFS. Random IFS contain deterministic IFS as a special case.

We prove under a mild assumption of finite conditional entropy, the dimensional exactness of the projections of invariant measures from the shift space, in the context of random conformal iterated function systems with countable alphabet and arbitrary overlaps. We thus deal simultaneously with two new, and qualitatively different issues: