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Weak amenability of Fourier algebras and local synthesis of the anti-diagonal

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ABSTRACT

We show that for a connected Lie group G, its Fourier algebra $A(G)$ is weakly amenable only if G is abelian. Our main new idea is to show that weak amenability of $A(G)$ implies that the anti-diagonal, $\check{\Delta}_G = \{(g, g^{-1}) : g \in G\}$, is a set of local synthesis for $A(G \times G)$. We then show that this cannot happen if G is non-abelian. We conclude for a locally compact group G, that $A(G)$ can be weakly amenable only if it contains no closed connected non-abelian Lie subgroups. In particular, for a Lie group G, $A(G)$ is weakly amenable if and only if its connected component of the identity G_e is abelian.

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0.1. Background

Questions on the nature of bounded derivations on (commutative) Banach algebras $\mathcal{A}$ have been around for a long time, in particular vanishing of bounded Hochschild cohomologies $H^1(\mathcal{A},\mathcal{M})$ for certain Banach $\mathcal{A}$-modules $\mathcal{M}$. See, for example, [38,19]. Johnson systematized many of these questions in [16]. In particular, he showed that for a locally compact group G, its group algebra is *amenable* (i.e. $H^1(L^1(G),\mathcal{M}^*)=\{0\}$ for each dual module $\mathcal{M}^*$) if and only if G is an amenable group. He also started the problem of determining when $H^1(L^1(G),L^1(G)^*)=\{0\}$.

For a commutative Banach algebra $\mathcal{A}$, Bade, Curtis and Dales [2] introduced the concept of *weak amenability*, which is defined as having $H^1(\mathcal{A},\mathcal{M})=\{0\}$ for all symmetric Banach modules. They observed that this is equivalent to having $H^1(\mathcal{A},\mathcal{A}^*)=\{0\}$. There is an interesting universal module also exhibited by Runde [35]. The above observation of [2], leads us to refer to any Banach algebra $\mathcal{B}$ as *weakly amenable* if $H^1(\mathcal{B},\mathcal{B}^*)=\{0\}$. Weak amenability was established for all $L^1(G)$ by Johnson [17].

The Fourier algebras, $A(G)$, as defined by Eymard [6], are dual objects to the group algebras $L^1(G)$ in a sense which generalizes Pontryagin duality. It was long expected that the amenability properties enjoyed by group algebras would also extend to Fourier algebras. Hence it was a surprise when Johnson [18] showed that $A(G)$ fails to be weakly amenable for any compact simple Lie group, in particular for $G=\mathrm{SO}(3)$. This motivated Ruan [34] to consider the operator space structure $A(G)$ inherits by virtue of being the predual of a von Neuman algebra. He proved that $A(G)$ is operator amenable if and only if G is amenable. Operator weak amenability for general $A(G)$ was determined by Spronk [39] and, independently, by Samei [36]. The question of amenability of $A(G)$ was settled by Forrest and Runde [8]: it happens exactly when G is virtually abelian. They also showed that if the connected component G_e is abelian, then $A(G)$ is weakly amenable. The following is suggested.

Question 0.1. *If $A(G)$ is weakly amenable, then must G_e be abelian?*

Much progress has been made in answering this question. Building on work of Plymen [32] – which was written to answer a question in [18] – Forrest, Samei and Spronk [9] showed that $A(G)$ is not weakly amenable whenever G contains a non-abelian connected compact subgroup. Exciting recent progress was made by Choi and Ghandehari. In [3] they show for the affine motion group, and hence any simply connected semisimple Lie group, and also for the reduced Heisenberg group, that the Fourier algebra is not

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