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Critical metrics on connected sums of Einstein four-manifolds

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ABSTRACT

We develop a gluing procedure designed to obtain canonical metrics on connected sums of Einstein four-manifolds. The main application is an existence result, using two well-known Einstein manifolds as building blocks: the Fubini–Study metric on $\mathbb{CP}^2$ and the product metric on $S^2 \times S^2$. Using these metrics in various gluing configurations, toric-invariant critical metrics are found on connected sums for a specific Riemannian functional, which depends on the global geometry of the factors. Furthermore, using certain quotients of $S^2 \times S^2$ as one of the gluing factors, critical metrics on several non-simply-connected manifolds are also obtained.

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1. Introduction

A Riemannian manifold (M^4, g) in dimension four is critical for the Einstein–Hilbert functional

$$\mathcal{R}(g) = Vol(g)^{-1/2} \int_M R_g dV_g, \tag{1.1}$$

where R_g is the scalar curvature if and only if it satisfies

$$Ric(g) = \lambda \cdot g, \tag{1.2}$$

where λ is a constant; such Riemannian manifolds are called *Einstein manifolds*. Non-collapsing limits of Einstein manifolds have been studied in great depth [3,7,37]. In particular, with certain geometric conditions, the limit space is an orbifold, with asymptotically locally Euclidean (ALE) spaces bubbling off at the singular points. A natural question is whether it is possible to reverse this process: Can one start with the limit space, and glue on a bubble in order to obtain an Einstein metric? A recent article of Olivier Biquard makes great strides in the Poincaré–Einstein setting [10]. In this work it is shown that a $\mathbb{Z}/2\mathbb{Z}$-orbifold singularity p of a non-degenerate Poincaré–Einstein orbifold (M, g) has a Poincaré–Einstein resolution obtained by gluing on an Eguchi–Hanson metric if and only if the condition

$$\det(\mathbf{R}^+(p)) = 0 \tag{1.3}$$

is satisfied, where $\mathbf{R}^+(p) : \Lambda^2_+ \to \Lambda^2_+$ is the purely self-dual part of the curvature operator at p. The self-adjointness of this gluing problem is overcome by the freedom of changing the boundary data of the Poincaré–Einstein metric.

However, not much is known about gluing compact manifolds together in the Einstein case. In this work, we will replace the Einstein equations with a generalization of the Einstein condition. Namely, we ask whether it is possible to glue together Einstein metrics and produce a critical point of a certain Riemannian functional generalizing