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General systems of linear forms: Equidistribution and true complexity

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ABSTRACT

Higher-order Fourier analysis is a powerful tool that can be used to analyze the densities of linear systems (such as arithmetic progressions) in subsets of Abelian groups. We are interested in the group F_p^n, for fixed p and large n, where it is known that analyzing these averages reduces to understanding the joint distribution of a family of sufficiently pseudorandom (formally, high-rank) nonclassical polynomials applied to the corresponding system of linear forms.

In this work, we give a complete characterization for these distributions for arbitrary systems of linear forms. This extends previous works which accomplished this in some special cases. As an application, we resolve a conjecture of Gowers and Wolf on the true complexity of linear systems. Our proof deviates from that of the previously known special cases and requires several new ingredients. One of which, which may be of independent interest, is a new theory of homogeneous nonclassical polynomials.

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1. Introduction

Gowers' seminal work in combinatorial number theory [5] initiated an extension of the classical Fourier analysis, called *higher-order Fourier analysis* of Abelian groups. Higher-order Fourier analysis has been very successful in dealing with problems regarding the densities of small linear structures (e.g. arithmetic progressions) in subsets of Abelian groups. It is possible to express such densities as certain analytic averages. For example, the density of the three term arithmetic progressions in a subset A of an Abelian group G can be expressed as $\mathbf{E}_{x,y\in G}\left[\mathbf{1}_A(x)\mathbf{1}_A(x+y)\mathbf{1}_A(x+2y)\right]$. More generally, one is often interested in analyzing

$$\mathop{\mathbf{E}}_{x_1,\dots,x_k\in G}\left[\mathbf{1}_A(L_1(x_1,\dots,x_k))\cdots\mathbf{1}_A(L_m(x_1,\dots,x_k))\right], \tag{1}$$

where each L_i is a linear form on k variables. Averages of this type are of interest in computer science, additive combinatorics, and analytic number theory.

In this paper we are only interested in the group $\mathbb{F}^n$ where $\mathbb{F}=\mathbb{F}_p$ for a fixed prime p and n is large. In the classical Fourier analysis of $\mathbb{F}^n$, a function is expressed as a linear combination of the characters of $\mathbb{F}^n$. Note that the characters of $\mathbb{F}^n$ are exponentials of linear polynomials: for $\alpha\in\mathbb{F}^n$, the corresponding character is defined as $\chi_\alpha(x)=\mathsf{e}\left(1/p\cdot\sum_{i=1}^n\alpha_i x_i\right)$, where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is the torus and $\mathsf{e}:\mathbb{T}\to\mathbb{C}$ is given by $\mathsf{e}(a)=\exp(2\pi i\cdot a)$. In higher-order Fourier analysis, the linear polynomials are replaced by higher degree "nonclassical polynomials" (a generalization of classical polynomials), and one would like to approximate a function $f:\mathbb{F}^n\to\mathbb{C}$ by a linear combination of such higher-order terms. A thorough treatment of nonclassical polynomials is presented in Section 2.1. For now, it suffices to say that a degree-d nonclassical polynomial of depth $k\geqslant 0$ is given by a function $P:\mathbb{F}^n\to\mathbb{U}_{k+1}$, where $\mathbb{U}_{k+1}=\frac{1}{p^{k+1}}\mathbb{Z}/\mathbb{Z}$ is a discrete subgroup of $\mathbb{T}$, such that P vanishes after taking $d+1$ additive derivatives. The case $k=0$ corresponds to classical polynomials. The existence of such approximations is a consequence of the so-called "inverse theorems" for Gowers norms which are established in a sequence of papers by Bergelson, Green, Samorodnitsky, Szegedy, Tao, and Ziegler [19,20,17,13,11,1,16].

Higher-order Fourier expansions are extremely useful in studying averages that are defined through linear structures. To analyze the average in (1), one approximates $1_A\approx\Gamma(P_1,\dots,P_C)$ where C is a constant which depends only on the approximation guarantees, $P_1,\dots,P_C$ are bounded degree polynomials of corresponding depths $k_1,\dots,k_C$, and $\Gamma:\prod_{i=1}^C\mathbb{U}_{k_i+1}\to\mathbb{R}$ is some composition function. Then applying the classical Fourier transform to Γ yields the higher-order Fourier expansion

$$\mathbf{1}_A\approx\Gamma(P_1,\dots,P_C)\approx\sum_\alpha\widehat{\Gamma}(\alpha)\,\mathsf{e}\left(\sum_{i=1}^C\alpha_i P_i\right),$$

where the coefficients $\widehat{\Gamma}(\alpha)$ are complex numbers, and α takes values in $\prod_{i=1}^C\mathbb{Z}_{p^{k_i+1}}\cong\prod_{i=1}^C\mathbb{U}_{k_i+1}$.