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Functional equations for double series of Euler type with coefficients

YoungJu Choie[a,*,1], Kohji Matsumoto[b]

[a] Department of Mathematics and PMI, Pohang University of Science and Technology, Pohang, 790–784, Republic of Korea
[b] Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan

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ABSTRACT

We prove two types of functional equations for double series of Euler type with complex coefficients. The first one is a generalization of the functional equation for the Euler double zeta-function, proved in a former work of the second-named author. The second one is more specific, which is proved when the coefficients are Fourier coefficients of cusp forms and the modular relation is essentially used in the course of the proof. As a consequence of functional equation we are able to determine trivial zero divisors.

* Corresponding author.
E-mail addresses: yjc@postech.ac.kr (Y. Choie), kohjimat@math.nagoya-u.ac.jp (K. Matsumoto).

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1. Introduction

Inspired by two sources, namely, the theory of multiple zeta values on the one hand, and the theory of modular symbols and periods of cusp forms on the other, Manin in [9,8] extended the theory of periods of modular forms replacing integration along geodesics in the complex upper half plane by iterated integrations to set up the foundation of the theory of "iterated non-commutative modular symbols". In particular Manin [9,8] considered the following iterated Mellin transform

$$I_{i\infty}^{0}(\omega_{s_\ell}, .., \omega_{s_1}) := \int_{i\infty}^{0} \omega_{s_\ell}(\tau_\ell) \int_{i\infty}^{\tau_\ell} \omega_{s_{\ell-1}}(\tau_{\ell-1}) .. \int_{i\infty}^{\tau_2} \omega_{s_1}(\tau_1)$$

of a finite sequence of cusp forms $f_1, \ldots, f_\ell$ of weight $k_j \in \mathbb{N}$ with respect to a congruence subgroup Γ of $SL_2(\mathbb{Z})$ and $\omega_{s_j}(\tau) := f_j(\tau)\tau^{s_j-1}d\tau, s_j \in \mathbb{C}, j = 1, \ldots, \ell$. When $\ell = 1$ then

$$I_{i\infty}^{0}(\omega_s) = \int_{i\infty}^{0} f(\tau)\tau^{s-1}d\tau$$

is the classical Mellin transform of a cusp form $f \in S_k(\Gamma)$ satisfying the following functional equation:

$$I_{i\infty}^{0}(\omega_s) = -\epsilon_f e^{\pi i s} N^{\frac{k}{2}-s} I_{i\infty}^{0}(\omega_{k-s})$$

if f is an eigenform with eigenvalue $\epsilon_f = \pm 1$ with respect to the involution $\omega_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ (see [17]).

However it seems that it is not anymore true to expect a simple functional equation if $\ell \geq 2$. Manin [8] said "since a neat functional equation can be written not for these individual integrals but for their generating series..," so the functional equation of the "total Mellin transform" associated to the finite family $\{f_j | j = 1, .., \ell, ..\}$ of cusp forms was derived.

Now consider the case when $\ell = 2$: for $f_j(\tau) = \sum_{n\geq 1} a_j(n)e^{2\pi i n\tau}, j = 1, 2$, we have

$$\begin{aligned} I_{i\infty}^{0}(\omega_{s_2}, \omega_{s_1}) &= \int_{i\infty}^{0} f_2(\tau_2){\tau_2}^{s_2-1} \int_{i\infty}^{\tau_2} f_1(\tau_1){\tau_1}^{s_1-1}d\tau_1 d\tau_2 \\ &= \int_{i\infty}^{0} f_2(\tau_2){\tau_2}^{s_2-1}d\tau_2 \int_{i\infty}^{0} f_1(\tau_1+\tau_2)(\tau_1+\tau_2)^{s_1-1}d\tau_1. \end{aligned}$$