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Homological stability for topological chiral homology of completions

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ABSTRACT

By proving that several new complexes of embedded disks are highly connected, we obtain several new homological stability results. Our main result is homological stability for topological chiral homology on an open manifold with coefficients in certain partial framed E_n-algebras. Using this, we prove a special case of a conjecture of Vakil and Wood on homological stability for complements of closures of particular strata in the symmetric powers of an open manifold and we prove that the bounded symmetric powers of closed manifolds satisfy homological stability rationally.

Contents

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1. Introduction

In this paper we prove a generalization of homological stability for configuration spaces. Let M be a manifold and let $C_k(M)$ denote the configuration space of k distinct unordered particles in M. If M is open, then there is a map $t : C_k(M) \to C_{k+1}(M)$ adding a particle near infinity; homological stability for configuration spaces says that this map is an isomorphism in homology in a range tending to infinity with k.

Our generalization involves certain configuration spaces with summable labels. For example, if A is a commutative monoid, then we consider spaces of particles in M with labels in A topologized such that if the particles collide we add their labels. However, such a construction makes sense in a more general setting; the labels only need to have the structure of a so-called *framed E_n-algebra* [33,15]. The analogous construction of the labeled configuration space is then called *topological chiral homology* [32] and is denoted $\int_M A$. It is also known as *factorization homology* [2] or *configuration spaces of particles with summable labels* [45].

We will consider framed E_n-algebras A with $\pi_0(A) = \mathbb{N}_0$ that are "generated by finitely many components," a notion made precise using *completions* of framed E_n-algebras. If $\pi_0(A) = \mathbb{N}_0$ the connected components of $\int_M A$ are in bijection with $\mathbb{N}_0$ for connected M and we denote the kth component by $\int_M^k A$. When M is open, there is again a stabilization map

$$t : \int_M^k A \to \int_M^{k+1} A.$$

The main result of this paper is that for A generated by finitely many components, this map induces an isomorphism in homology in a range tending to infinity with k.

1.1. Topological chiral homology

We start by introducing the spaces we are interested in, which are defined using topological chiral homology.

Topological chiral homology is a homology theory for n-dimensional manifolds. When discussing topological chiral homology, one needs to fix a background symmetric monoidal $(\infty, 1)$-category like chain complexes or spectra. In this paper, this category will always be taken to be Top, the category of topological spaces with Cartesian product.