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Generalized translation invariant valuations and the polytope algebra ☆

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A B S T R A C T

We study the space of generalized translation invariant valuations on a finite-dimensional vector space and construct a partial convolution which extends the convolution of smooth translation invariant valuations. Our main theorem is that McMullen's polytope algebra is a subalgebra of the (partial) convolution algebra of generalized translation invariant valuations. More precisely, we show that the polytope algebra embeds injectively into the space of generalized translation invariant valuations and that for polytopes in general position, the convolution is defined and corresponds to the product in the polytope algebra.

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1. Introduction

Let V be an n-dimensional vector space, V^* the dual vector space, $\mathcal{K}(V)$ the set of non-empty compact convex subsets in V, endowed with the topology induced by the Hausdorff metric for an arbitrary Euclidean structure on V, and $\mathcal{P}(V)$ the set of polytopes in V. A valuation is a map $\mu : \mathcal{K}(V) \to \mathbb{C}$ such that

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$. Continuity of valuations will be with respect to the Hausdorff topology.

Examples of valuations are measures, the intrinsic volumes (in particular the Euler characteristic χ) and mixed volumes.

Let $\mathrm{Val}(V)$ denote the (Banach-)space of continuous, translation invariant valuations. It was the object of intensive research during the last few years, compare [3,9,11,12,14–17,19,21] and the references therein.

Valuations with values in semi-groups other than $\mathbb{C}$ have also attracted a lot of interest. We only mention the recent papers [1,2,18,23,25,29,31–34] to give a flavor on this active research area.

Of particular importance is the class of the so-called smooth valuations because it admits various algebraic structures, which include two bilinear pairings, known as product and convolution, and a Fourier-type duality interchanging them. These algebraic structures are closely related to important notions from convex and integral geometry, such as the Minkowski sum, mixed volumes, and kinematic formulas. This emerging new theory is known as algebraic integral geometry [14,21].

A different, more classical type of algebraic object playing an important role in convex geometry is McMullen's algebra of polytopes. In this paper, we show how McMullen's algebra fits into the framework of algebraic integral geometry. More precisely, we show that McMullen's algebra can be embedded as a subalgebra of the space of generalized valuations, which is, roughly speaking, the dual space of smooth valuations.

Let us now give the necessary background required to state our main theorems.

The group $\mathrm{GL}(V)$ acts in the natural way on $\mathrm{Val}(V)$. The dense subspace of $\mathrm{GL}(V)$-smooth vectors in $\mathrm{Val}(V)$ is denoted by $\mathrm{Val}^\infty(V)$. It carries a Fréchet topology which is finer than the induced topology.

In [16], a convolution product on $\mathrm{Val}^\infty(V) \otimes \mathrm{Dens}(V^*)$ was constructed. Here and in the following, $\mathrm{Dens}(W)$ denotes the 1-dimensional space of densities on a linear space W. Note that $\mathrm{Dens}(V) \otimes \mathrm{Dens}(V^*) \cong \mathbb{C}$: if vol is any choice of Lebesgue measure on V, and vol^* the corresponding dual measure on V^*, then $\mathrm{vol} \otimes \mathrm{vol}^* \in \mathrm{Dens}(V) \otimes \mathrm{Dens}(V^*)$ is independent of the choice of vol. If $\phi_i(K) = \mathrm{vol}(K + A_i) \otimes \mathrm{vol}^*$ with smooth compact strictly convex bodies A_1, A_2, then $\phi_1 * \phi_2(K) = \mathrm{vol}(K + A_1 + A_2) \otimes \mathrm{vol}^*$. By Alesker's proof [3] of McMullen's conjecture, linear combinations of such valuations are dense in