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Singular integrals, maximal functions and Fourier restriction to spheres: The disk multiplier revisited

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ABSTRACT

Several estimates for singular integrals, maximal functions and spherical summation operator are given in the spaces $L^p_{\text{rad}}L^2_{\text{ang}}(\mathbb{R}^n)$, $n \geq 2$.

1. Introduction

A well-known open problem in Fourier analysis is the Bochner–Riesz operator conjecture, which asserts the L^p-boundedness of the Fourier multipliers

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$$\widehat{T_\alpha f}(\xi) = \left(1 - |\xi|^2\right)_+^\alpha \widehat{f}(\xi)$$

on $L^p(\mathbb{R}^n)$, so long as

$$\frac{2n}{n+1+2\alpha} < p < \frac{2n}{n-1-2\alpha}$$

where $0 < \alpha < (n-1)/2$ and

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\, dx$$

denotes the Fourier transform in $\mathbb{R}^n$.

The problem is well understood in dimensions $n = 1$ and $n = 2$ (see [27,6,15,9]). But in higher dimensions, although there are several interesting results by many authors, it remains open.

Its relevance is due, on the one hand, to the very natural question being asked, but also because of its close connection with some other basic objects, namely the so-called Kakeya maximal function, the restriction properties of the Fourier transform, or the covering properties satisfied by parallelepipeds in $\mathbb{R}^n$ having arbitrary directions and eccentricities.

There is also the hope that obtaining deep understanding of the Bochner–Riesz operators could be a first step in the project of extending the classical Calderón–Zygmund theory of singular integrals, or pseudodifferential operators, going beyond kernels whose singularities are located only at the origin or at infinity, as is demanded in several areas of number theory or PDEs.

In the extreme case, $\alpha = 0$, the multiplier $T = T_0$ is given by the indicator function of the unit ball. By a remarkable result of C. Fefferman [14] we know that it is bounded only in the obvious case $p = 2$, disproving the conjecture about the boundedness of T in the range $2n/(n+1) < p < 2n/(n-1)$.

In its proof Fefferman made use of the properties of the Kakeya sets in the plane (for every $N \gg 1$ there is a set whose measure is less than $1/\log N$ but containing a rectangle of dimensions $1 \times 1/N$ on every direction), but also of a previous result due to Y. Meyer, who observed that the L^p-boundedness of T implies a vector-valued control for Hilbert transforms in different directions of the space. More concretely:

Let

$$H_\omega f(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f(x - t\omega)}{t}\, dt, \qquad \omega \in S^{n-1}. \tag{1}$$

Then T bounded on $L^p(\mathbb{R}^n)$ implies

$$\left\| \left(\sum |H_{\omega_j} f_j|^2 \right)^{1/2} \right\|_p \lesssim \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_p.$$