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Poincaré series for tensor invariants and the McKay correspondence

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ABSTRACT

For a finite group G and a finite-dimensional G-module V, we prove a general result on the Poincaré series for the G-invariants in the tensor algebra $\mathsf{T}(\mathsf{V}) = \bigoplus_{k\geq 0} \mathsf{V}^{\otimes k}$. We apply this result to the finite subgroups G of the 2×2 special unitary matrices and their natural module V of 2×1 column vectors. Because these subgroups are in one-to-one correspondence with the simply laced affine Dynkin diagrams by the McKay correspondence, the Poincaré series obtained are the generating functions for the number of walks on the simply laced affine Dynkin diagrams.

1. Introduction

Let G be a group, and assume $\{\mathsf{G}^\lambda \mid \lambda \in \Lambda(\mathsf{G})\}$ is the set of finite-dimensional irreducible G-modules over the complex field $\mathbb{C}$. Associated to a fixed finite-dimensional

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G-module V over $\mathbb{C}$ is the *representation graph* $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ having nodes indexed by the λ in $\Lambda(\mathsf{G})$ and $a_{\mu,\lambda}$ edges from μ to λ in $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ if

$$\mathsf{G}^{\mu} \otimes \mathsf{V} = \bigoplus_{\lambda \in \Lambda(\mathsf{G})} a_{\mu,\lambda} \mathsf{G}^{\lambda}. \tag{1.1}$$

Thus, the number of edges $a_{\mu,\lambda}$ from μ to λ in $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ is the multiplicity of G^{λ} as a summand of $\mathsf{G}^{\mu} \otimes \mathsf{V}$.

When G is a finite group, the representation graph and the characters of G are closely related. Assume χ_{V} is the character of V, and χ_{λ} is the character of G^{λ} for $\lambda \in \Lambda(\mathsf{G})$. Let $\mathsf{d} = \dim \mathsf{V} = \chi_{\mathsf{V}}(1)$. Steinberg [31] has shown that when the action of G on V is faithful, the following hold:

- The eigenvalues of the matrix $\left(\mathsf{d}\,\delta_{\mu,\lambda} - a_{\mu,\lambda}\right)$ are $\mathsf{d} - \chi_{\mathsf{V}}(g)$ as g ranges over a set Γ of conjugacy class representatives of G.
- The column vector $\left(\chi_{\lambda}(g)\right)$ with entries given by the character values of the irreducible G-modules at g is an eigenvector corresponding to $\mathsf{d} - \chi_{\mathsf{V}}(g)$. These vectors form the columns of the character table of G.
- The vector $\left(\mathsf{d}^{\lambda}\right)$, whose entries are the dimensions $\mathsf{d}^{\lambda} = \dim \mathsf{G}^{\lambda} = \chi_{\lambda}(1)$ of the irreducible G-modules, corresponds to the eigenvalue 0.

Let G^0 be the one-dimensional trivial G-module on which every element of the group acts as the identity transformation, and let m_k^{λ} be the number of walks of k steps from 0 to λ on the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$. Since each step on the graph is accomplished by tensoring with V, m_k^{λ} is the multiplicity of the irreducible G-module G^{λ} in $\mathsf{G}^0 \otimes \mathsf{V}^{\otimes k} \cong \mathsf{V}^{\otimes k}$.

In what follows, we identify $\mathsf{V}^{\otimes 0} \cong \mathbb{C}$ as a G-module with G^0, so that $\mathsf{m}_0^{\lambda} = \delta_{\lambda,0}$ (the Kronecker delta). For $\lambda \in \Lambda(\mathsf{G})$, we consider the Poincaré series

$$\mathsf{m}^{\lambda}(t) = \sum_{k \geq 0} \mathsf{m}_k^{\lambda}\, t^k \tag{1.2}$$

for the multiplicity of G^{λ} in the tensor algebra $\mathsf{T}(\mathsf{V}) = \bigoplus_{k \geq 0} \mathsf{V}^{\otimes k}$ (which is also the generating function for the number of walks from 0 to λ in $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$). In particular, $\mathsf{m}^0(t)$ is the Poincaré series for the G-invariants, $\mathsf{T}(\mathsf{V})^{\mathsf{G}} = \{w \in \mathsf{T}(\mathsf{V}) \mid gw = w \text{ for all } g \in \mathsf{G}\}$, in $\mathsf{T}(\mathsf{V})$.

The *centralizer algebra*,

$$\mathsf{Z}_k(\mathsf{G}) = \{X \in \mathsf{End}(\mathsf{V}^{\otimes k}) \mid Xgw = gXw \ \text{ for all } \ w \in \mathsf{V}^{\otimes k}\},$$

plays an essential role in understanding the G-module $\mathsf{V}^{\otimes k}$. The idempotents that project $\mathsf{V}^{\otimes k}$ onto its irreducible G-summands live in the finite-dimensional semisimple associative algebra $\mathsf{Z}_k(\mathsf{G})$. *Schur–Weyl duality* relates the decomposition of $\mathsf{V}^{\otimes k}$ as a G-module to