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Cubic modular equations in two variables

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ABSTRACT

By adding certain equianharmonic elliptic sigma functions to the coefficients of the Borwein cubic theta functions, an interesting set of six two-variable theta functions may be derived. These theta functions invert the $F_1\left(\frac{1}{3};\frac{1}{3};\frac{1}{3};1|x,y\right)$ case of Appell's hypergeometric function and satisfy several identities akin to those satisfied by the Borwein cubic theta functions. The work of Koike et al. is extended and put into the context of modular equations, resulting in a simpler derivation of their results as well as several new modular equations for Picard modular functions. An application of these results is a new two-parameter family of solvable nonic equations.

1. Introduction

The F_1 function is defined for $|x| < 1$ and $|y| < 1$ by

$$F_1\left(a;b_1;b_2;c|x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{x^m y^n}{m!\,n!}, \tag{1.1}$$

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where $(a)_n = a(a+1)\cdots(a+n-1)$, and an analytic continuation for $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(c-a) > 0$ is given by the integral representation

$$F_1\left(a; b_1; b_2; c|x, y\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}dt. \quad (1.2)$$

In [13] the striking identity

$$F\left(1-x^3, 1-y^3\right) = \frac{3}{1+x+y} F\left(\left(\frac{1+\omega x+\bar{\omega} y}{1+x+y}\right)^3, \left(\frac{1+\bar{\omega} x+\omega y}{1+x+y}\right)^3\right) \quad (1.3)$$

was derived in connection with the common limit of a three term iteration. Here, $\omega = e^{2\pi i/3}$, $F(x, y) = F_1\left(\frac{1}{3}; \frac{1}{3}; \frac{1}{3}; 1|x, y\right)$, and the function F_1 as defined in (1.1) is the first of the four two-variable hypergeometric functions introduced by Appell [1]. We will show that such an identity is part of a larger class of identities and derive the next member of this class,

$$F\left(\frac{x^3(y^2+3)(xy^2-3x-6y)}{(xy-3)^3(xy+3)}, \frac{y^3(x^2+3)(yx^2-3y-6x)}{(xy-3)^3(xy+3)}\right) = \frac{xy-3}{xy-3x-3y-3}$$
$$\times F\left(\frac{(x^2+3)(y+3)^3(yx^2-3y-6x)}{(xy+3)(xy-3x-3y-3)^3}, \frac{(y^2+3)(x+3)^3(xy^2-3x-6y)}{(xy+3)(xy-3x-3y-3)^3}\right). \quad (1.4)$$

Due to the reduction formula

$$F_1(a; b_1, b_2; c|x, x) = {}_2F_1\left(a, b_1 + b_2; c|x\right),$$

the specialization of $x = y$ in (1.3) and (1.4) reduces them to transformations involving the one-variable function

$$F(x) = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1|x\right).$$

Thus, (1.3) and (1.4) can be viewed as two-variable generalizations of modular equations arising from Ramanujan's theory of elliptic functions to base three (theory of signature three). For this reason, we first review several of the key results of this theory in Section 2 before stating the main results on two-variable generalizations in Section 3. Sections 4 and 5 introduce six Θ constants that are central to obtaining (1.3) and (1.4). Finally, Section 6 gives motivated and simple proofs of (1.3) and (1.4) based on identities of Θ functions, while Section 7 gives some applications of the modular equations contained in (1.3) and (1.4).

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