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# Logical metatheorems for abstract spaces axiomatized in positive bounded logic

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ABSTRACT

In this paper we show that normed structures which can be axiomatized in positive bounded logic (in the sense of Henson and Iovino) admit proof-theoretic metatheorems (as developed by the second author since 2005) on the extractability of explicit uniform bounds from proofs in the respective theories. We apply this to design such metatheorems for abstract Banach lattices, $L^p$- and $C(K)$-spaces as well as bands in $L^p(L^q)$-Bochner spaces. We also show that a proof-theoretic uniform boundedness principle can serve in many ways as a substitute for the model-theoretic use of ultrapowers of Banach spaces.



## 1. Introduction

During the last decade, proof-theoretic results (so-called logical metatheorems due to the second author) have been developed which allow one to extract finitary computational content in the form of explicit uniform bounds from prima facie noneffective proofs in abstract nonlinear analysis (see [29] and the subsequent extensions in [16]

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and [31] as well as [34,33,25,32,35] for some recent applications). 'Abstract' here refers to the fact that the proofs analyzed concern general classes of metric structures $X$ (in addition to concrete structures such as $\mathbb{R}$ or $C[0,1]$ whose proof-theoretic treatment is covered already by e.g. [28]). As the proof-theoretic methods used in this context are based on extensions and variants of Gödel's functional ('Dialectica') interpretation, the basic condition on the classes of structures to be admissible is that they can be axiomatized by axioms having a (simple) computable solution of their (monotone) functional interpretation (given enrichments by suitable moduli e.g. of uniform convexity, uniform smoothness etc.). Structures treated so far include metric and normed spaces and their completions, $W$-hyperbolic spaces and CAT(0)-spaces, uniformly convex normed and hyperbolic spaces, uniformly smooth spaces, compact metric spaces. Notably absent in this list are the classes of smooth (but in general not uniformly smooth) or strictly convex (but in general not uniformly convex), separable (but in general not boundedly compact and hence not finite dimensional) normed spaces, incomplete metric spaces etc. These are classes of structures which are not closed under taking ultrapowers (w.r.t. a nonprincipal ultrafilter) of a normed (or metric) structure, since e.g. an ultrapower of a Banach space $X$ is strictly convex iff $X$ is uniformly convex. This already indicates a first point of connection between the proof-theoretic approach to metric and normed structures and the model theory of such structures as developed in the framework of continuous logic (due to [10], adapted by [6]) or positive bounded logic [19].

The proof-theoretic metatheorems referred to above and adapted to new classes of spaces in this paper, have the form to guarantee the extractability of explicit uniform effective bounds from proofs of large classes of statements provided that the proof can be formalized in a suitable formal framework (the permitted frameworks are so strong that this is no restriction in practice). The complexity and – in particular – the growth of the extracted bound reflect the computational content of the given proof. The metatheorems are applied via specialized formats adapted to the concrete situation at hand (see Corollaries 17.54, 17.55, 17.59, 17.70, 17.71 in [31]). E.g. we may have some iterative procedure $(x_n)$ (e.g. the Krasnoselski, Mann, Ishikawa, Halpern or Bruck iteration) based on a map $T : X \to X$ starting from some $x_0 \in X$ considered for which either asymptotic regularity results of the form $\|x_n - T(x_n)\| \to 0$ or strong convergence results for $(x_n)$ can be proven. Then the metatheorems can be applied to extract rates of asymptotic regularity resp. of metastability in the sense of T. Tao which, as far as $x_0, X, T$ are concerned, only depend on a bound $\mathbb{N} \ni b \geq \|x_0\|, \|x_0 - T(x_0)\|$ and some so-called majorant $T^*$ for $T$. In the important case (both in fixed point theory as well as ergodic theory) where $T$ is nonexpansive, $T^*$ can be defined as $T^*(n) := n + 3b$ (see [31, p. 419]). This approach has been applied to fixed point theory, ergodic theory, topological dynamics, geodesic geometry, convex optimization, image recovery problems and abstract Cauchy problems in more than 50 papers during the last decade. A survey of applications up to 2008 can be found in [31]. For some more recent applications of logical metatheorems (and related techniques) see e.g. [2,4,34,33,32,35].