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On a problem of Bousfield for metabelian groups ☆

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ABSTRACT

The homological properties of localizations and completions of metabelian groups are studied. It is shown that, for $R = \mathbb{Q}$ or $R = \mathbb{Z}/n$ and a finitely presented metabelian group G, the natural map from G to its R-completion induces an epimorphism of homology groups $H_2(-, R)$. This answers a problem of A.K. Bousfield for the class of metabelian groups.

1. Introduction

The subject of investigation of this paper is the relation between the inverse limits of groups and the second homology $H_2(-, K)$ for certain coefficients K. One of the results of the paper is the following. Let G be a finitely presented metabelian group, $\{\gamma_i(G)\}_{i\geq 0}$ its lower central series. Then, for any $n > 0$, there is a natural isomorphism

$$H_2(\varprojlim G/\gamma_i(G), \mathbb{Z}/n) \simeq \varprojlim H_2(G/\gamma_i(G), \mathbb{Z}/n).$$

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That is, in this particular case, the inverse limit commutes with the second homology functor.

The problem relating the inverse limit to the second homology of groups appears in different areas of algebra and topology. Recall two related open problems, one from [6], the second from [12]. A.K. Bousfield posed the following question in [6, Problem 4.10]:

Problem *(Bousfield).* Is $E^R X \to \hat{X}_R$ an isomorphism when X is a finitely presented group and $R = \mathbb{Q}$ or $R = \mathbb{Z}/n$?

In the above problem, E^R is the HR-localization functor defined in [6] and $\hat{X}_R$ is the R-completion of the group X. It follows from the properties of HR-localization and E^R given in [6], that, for a finitely presented group X, the map $E^R X \to \hat{X}_R$ is an isomorphism if and only if the completion map $X \to \hat{X}_R$ induces an epimorphism $H_2(X, R) \to H_2(\hat{X}_R, R)$. In this paper we prove the following (see Corollary 9.2)

Theorem. *For a finitely presented metabelian group X, the natural map $E^R X \to \hat{X}_R$ is an isomorphism for $R = \mathbb{Q}$ or $R = \mathbb{Z}/n$.*

Observe that, as shown in [6], the above result cannot be generalized to the case $R = \mathbb{Z}$ as the following simple example shows. For the Klein bottle group $G = \langle a, b \mid a^{-1}bab = 1\rangle$, the second homology $H_2(\hat{G}, \mathbb{Z})$ is isomorphic to the exterior square of the 2-adic integers and therefore is uncountable.

The inverse limit of a tower of discrete groups comes with two distinct topologies, the discrete topology, and the inverse limit topology. This latter topology views the inverse limit as a subspace of the product of the groups in the tower. One can ask whether both the group homology of this discrete group and the continuous homology in this alternative topology agree. For any pro-p-group P, there are natural comparison maps between these homology groups (see [12] for detailed discussion):

$$\phi_n : H_n^{discrete}(P, \mathbb{Z}/p) \to H_n^{cont}(P, \mathbb{Z}/p).$$

Analogously, there are maps between cohomology groups $\phi^n : H^n_{cont}(P, \mathbb{Z}/p) \to H^n_{discrete}(P, \mathbb{Z}/p)$. In [12], G.A. Fernandez-Alcober, I.V. Kazachkov, V.N. Remeslennikov, and P. Symonds asked the following question.

Problem. Does there exist a finitely presented pro-p group P for which $\phi^2 : H^2_{\mathrm{cont}}(P, \mathbb{Z}/p) \to H^2_{\mathrm{discrete}}(P, \mathbb{Z}/p)$ is not an isomorphism?

See [22] for background on the continuous cohomology of pro-p groups. It is shown in [12] that, for a finitely presented pro-p-group P, the following two conditions are equivalent:

(1) the map $\phi^2 : H^2_{\mathrm{cont}}(P, \mathbb{Z}/p) \to H^2_{\mathrm{discrete}}(P, \mathbb{Z}/p)$ is an isomorphism;