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Existence of optimal ultrafilters and the fundamental complexity of simple theories

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A B S T R A C T

In the first edition of *Classification Theory*, the second author characterized the stable theories in terms of saturation of ultrapowers. Prior to this theorem, stability had already been defined in terms of counting types, and the unstable formula theorem was known. A contribution of the ultrapower characterization was that it involved sorting out the global theory, and introducing nonforking, seminal for the development of stability theory. Prior to the present paper, there had been no such ultrapower characterization of an unstable class. In the present paper, we first establish the existence of so-called optimal ultrafilters on (suitable) Boolean algebras, which are to simple theories as Keisler's good ultrafilters [15] are to all (first-order) theories. Then, assuming a supercompact cardinal, we characterize the simple theories in terms of saturation of ultrapowers. To do so, we lay the groundwork for analyzing the global structure of simple theories, in ZFC, via complexity of certain amalgamation patterns. This brings into focus

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a fundamental complexity in simple unstable theories having no real analogue in stability.

1. Introduction

1.1. Background

We begin by giving some history and context of the power of ultraproducts as a tool in mathematics, and specifically in model theory. Ultrafilters on an infinite cardinal λ are maximal (under inclusion) subsets of the power set of λ which are closed under finite intersection, upward closed, and do not contain the empty set. These give a robust notion of largeness, allowing for infinite averaging arguments and the study of asymptotic or pseudofinite behavior in models. Early appearances were in the work of Tarski 1930 [35] on measures and Cartan 1937 [4,5] in general topology. The groundwork for their use in model theory was laid in the 1950s and early 1960s by Łoś [20], Tarski, Keisler [14], Frayne, Morel, and Scott [10], and Kochen [18] in terms of the *ultraproduct* construction. Given an ultrafilter $\mathcal{D}$ on λ, the ultraproduct N of a sequence of models $\langle M_\alpha : \alpha < \lambda \rangle$ in a fixed language $\mathcal{L}$ has as its domain the set of equivalence classes of elements of the Cartesian product $\prod_{\alpha<\lambda} M_\alpha$ under the equivalence relation of being equal on a set in $\mathcal{D}$. One then defines the relations, functions, and constants of $\mathcal{L}$ on each tuple of elements of the ultraproduct to reflect the average behavior across the index models. The fundamental theorem of ultraproducts, Łoś' theorem, says that the set of statements of first order logic true in the ultraproduct are precisely the statements true in a $\mathcal{D}$-large set of index models, i.e. the theory of N is the average theory of the models M_α. Model theorists concentrated further on so-called regular ultrafilters, as will be explained in due course.

This construction gave rise to some remarkable early transfer theorems. For example, Ax and Kochen [1–3] and independently Eršov [8] proved that for any nonprincipal (= containing all cofinite sets) ultrafilter $\mathcal{D}$ on the set of primes, the ultraproduct $\mathcal{Q}_p = \prod_p Q_p/\mathcal{D}$ of the p-adic fields Q_p and the ultraproduct $\mathcal{S}_p = \prod_p \mathbb{F}_p((t))/\mathcal{D}$ of the fields of formal power series over $\mathbb{F}_p$ are elementarily equivalent, i.e. satisfy the same first-order statements. Then from Lang's theorem that every homogeneous polynomial of degree $> d$ with more than d^2 variables has a nontrivial zero in $\mathbb{F}_p((t))$ for each p they deduce the corresponding theorem in Q_p for all but finitely many p.

Working with ultrapowers, meaning that all the index models are the same, a similar averaging process happens. The central "algebraic characterization of elementary equivalence" now appears: two models satisfy the same set of first order statements precisely when they have isomorphic ultrapowers, proved by Keisler 1961 under GCH [14] and by Shelah 1971 [32] in general.