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Congruence properties for a certain kind of partition functions

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A B S T R A C T

In light of the modular equations of fifth and seventh order, we derive some congruence properties for a certain kind of partition functions $a(n)$ which satisfy $\sum_{n=0}^{\infty} a(n)q^n \equiv (q;q)_\infty^k \pmod{m}$, where k is a positive integer with $1 \leq k \leq 24$ and $m = 2, 3$. In view of these properties, we obtain many infinite families of congruences for $c\phi_k(n)$, the number of generalized Frobenius partitions of n with k colors, and $\overline{c\phi_k}(n)$, the number of generalized Frobenius partitions of n with k colors whose order is k under cyclic permutation of the k colors. Meanwhile, we also apply the main theorems to some other kinds of partition functions.

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1. Introduction

The object of this paper is to discuss congruence properties for a certain kind of partition functions $a(n)$ which satisfy

$$\sum_{n=0}^{\infty} a(n)q^n \equiv f^k(-q) \pmod{m}, \tag{1.1}$$

where k and m are positive integers. Here we use the following notations

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad f(-q) := (q;q)_\infty.$$

By means of the modular equations of fifth and seventh order, we find a general method to obtain some infinite families of congruences for $a(n)$. In this paper, we mainly focus on the cases for $1 \le k \le 24$ and $m = 2, 3$. For large k and other values of m, we can also use the method to search for congruences for $a(n)$. The main results are stated as follows.

Theorem 1.1. *If $\sum_{n=0}^{\infty} a(n)q^n \equiv f^k(-q) \pmod 2$ with $1 \le k \le 24$, then for $\alpha \ge 1$, $n \ge 0$, and $i = 1, 2, 3, 4$, we have*

$$a\left(5^{t\alpha}n + \frac{(24i+5k)\cdot 5^{t\alpha-1}-k}{24}\right) \equiv 0 \pmod 2,$$

where

$$t = \begin{cases} 2, & k = 1,2,3,4,6,7,8,9,12,13,14,16,18,19,24, \\ 4, & \textit{otherwise}. \end{cases}$$

Theorem 1.2. *If $\sum_{n=0}^{\infty} a(n)q^n \equiv f^k(-q) \pmod 3$ with $1 \le k \le 24$, then for $\alpha \ge 1$, $n \ge 0$, and $i = 1, 2, 3, 4$, we have*

$$a\left(5^{t\alpha}n + \frac{(24i+5k)\cdot 5^{t\alpha-1}-k}{24}\right) \equiv 0 \pmod 3,$$

where

$$t = \begin{cases} 2, & \textit{if } k = 1,2,3,4,6,8,9,12,14,18,24, \\ 4, & \textit{if } k = 5,7,11,13,15,17,19,21,23, \\ 6, & \textit{otherwise}. \end{cases}$$