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Some supercongruences occurring in truncated hypergeometric series

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ABSTRACT

For the purposes of this paper *supercongruences* are congruences between terminating hypergeometric series and quotients of p-adic Gamma functions that are stronger than those one can expect to prove using commutative formal group laws. We prove a number of such supercongruences by using classical hypergeometric transformation formulae. These formulae, most of which are decades or centuries old, allow us to write the terminating series as the ratio of products of Γ-values. At this point *sums* have become *quotients*. Writing these Γ-quotients as Γ_p-quotients, we are in a situation that is well-suited for proving p-adic congruences. These Γ_p-functions can be p-adically approximated by their Taylor series expansions. Sometimes there is cancellation of the lower order terms, leading to stronger congruences. Using this technique we prove, among other things, a conjecture of Kibelbek and a strengthened version of a conjecture of van Hamme.

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1. Introduction

Set $(a)_k = a(a+1)\cdots(a+k-1) = \dfrac{\Gamma(a+k)}{\Gamma(a)}$, the rising factorial or the Pochhammer symbol, where $\Gamma(x)$ is the Gamma function. For r a nonnegative integer and $\alpha_i, \beta_i \in \mathbb{C}$, the (generalized) hypergeometric series ${}_{r+1}F_r$ defined by

$${}_{r+1}F_r\left[\begin{matrix}\alpha_1 & \dots & \alpha_{r+1} \\ & \beta_1 & \dots & \beta_r\end{matrix} ;\lambda\right] := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k\dots(\alpha_{r+1})_k}{(\beta_1)_k\dots(\beta_r)_k}\cdot\frac{\lambda^k}{k!}$$

converges for $|\lambda| < 1$ if it is well-defined. When any of the α_i is a negative integer and none of the β_i are negative integers larger than all α_j, the above sum terminates. We set

$${}_{r+1}F_r\left[\begin{matrix}\alpha_1 & \dots & \alpha_{r+1} \\ & \beta_1 & \dots & \beta_r\end{matrix} ;\lambda\right]_n := \sum_{k=0}^{n} \frac{(\alpha_1)_k(\alpha_2)_k\dots(\alpha_{r+1})_k}{(\beta_1)_k\dots(\beta_r)_k}\cdot\frac{\lambda^k}{k!},$$

the truncation of the series after the λ^n term.

Hypergeometric series are of fundamental importance in many research areas including algebraic varieties, differential equations, Fuchsian groups and modular forms. For instance, periods of algebraic varieties such as elliptic curves, certain K3 surfaces and other Calabi–Yau manifolds can be described by hypergeometric series [6]. Indeed, the Euler integral representation of ${}_2F_1$ (Theorem 2.2.1 of [4])

$$p_{\{a,b;c\}}(\lambda) := \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-\lambda t)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\cdot {}_2F_1\left[\begin{matrix}a & b \\ & c\end{matrix} ;\lambda\right] \quad (1.1)$$

holds when $\Re(c) > \Re(b) > 0$ and $\lambda \in \mathbb{C}\backslash[1,\infty)$. When $a = b = \frac{1}{2}$ and $c = 1$ the right side $p_{\{\frac{1}{2},\frac{1}{2};1\}}(\lambda)$ is a period of the Legendre family of elliptic curves $E_\lambda : y^2 = x(x-1)(x-\lambda)$ parameterized by λ. It is known that for general $a, b, c \in \mathbb{Q}$ under some suitable assumptions $p_{\{a,b;c\}}(\lambda)$ is a period of a generalized Legendre curve of the form $y^N = x^i(1-x)^j(1-\lambda x)^k$ where i, j, k can be computed from a, b, c, see [13]. It is worth mentioning that there are finite field analogues of hypergeometric series, in particular, the ${}_2F_1$ finite field analogues due to Greene can be used to compute the Galois representations of the generalized Legendre curves. For more information, see [13]. A generalization of the Euler integral representation to ${}_{r+1}F_r$ is presented in [27, §4.1]. These periods are in general complicated transcendental numbers. They are much more predictable when the elliptic curve has complex multiplication (CM), e.g. $\lambda = -1$. The Selberg–Chowla formula predicts that any period of a CM elliptic curve is an algebraic multiple of a quotient of Gamma values [26]. For instance, using a formula of Kummer (see (3.9)) one can compute that $p_{\{\frac{1}{2},\frac{1}{2};1\}}(-1) = \frac{\sqrt{2}}{4}\frac{\Gamma\left(\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{2}\right)}$. Similarly, $\pi^2 \cdot {}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1\end{matrix} ;\lambda\right]$ is a