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# Diophantine analysis in beta-dynamical systems and Hausdorff dimensions ☆

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ABSTRACT

Let $\{x_n\}_{n\geq 1} \subset [0,1]$ be a sequence of real numbers and let $\varphi\colon \mathbb{N} \to (0,1]$ be a positive function. Using the mass transference principle established by Beresnevich and Velani [1], we prove that for any $x \in (0,1]$, the Hausdorff dimension of the set

$$\{\beta > 1\colon |T_\beta^n x - x_n| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

satisfies a so-called 0–1 law according to $\limsup_{n\to\infty} \frac{\log \varphi(n)}{n} = -\infty$ or not, where $T_\beta$ is the $\beta$-transformation.


## 1. Introduction

Given a real number $\beta > 1$, the $\beta$-transformation $T_\beta\colon [0,1] \to [0,1]$ is defined by

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$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor \quad \text{for all } x \in [0,1],$$

where $\lfloor\cdot\rfloor$ denotes the integral part of a real number. The transformation $T_\beta$ on $[0,1]$ is a typical example of monotone one-dimensional expanding dynamical system. In 1957, Rényi [9] introduced this kind of map as a model for expanding a real number in a non-integer base $\beta > 1$. Since then, much attention has been paid to this transformation, see [2,4,6,10], etc.

For $\beta > 1$, the transformation $T_\beta$ has an invariant ergodic measure $\nu_\beta$, which is equivalent to the Lebesgue measure $\mathcal{L}$ on $[0,1]$ with the jump function

$$h_\beta(x) = \Theta(\beta) \sum_{x < T_\beta^n 1} \frac{1}{\beta^n}, \quad x \in [0,1]$$

as its density [7] and $\Theta(\beta)$ as the normalizing factor. This measure is the unique measure of maximal entropy [5], called the Parry measure.

Given a point $x \in (0,1]$, its orbits under $\beta$-transformations may have completely different distributions on $[0,1]$ when $\beta$ varies. For example, when $x = 1$, Blanchard [2] provided a classification of the parameter space $\{\beta \in \mathbb{R} \colon \beta > 1\}$ according to the distributions of the orbits $\mathcal{O}_\beta := \{T_\beta^n 1 \colon n \geq 1\}$:

Class $C_1$: $\mathcal{O}_\beta$ is ultimately zero.
Class $C_2$: $\mathcal{O}_\beta$ is ultimately non-zero periodic.
Class $C_3$: $\mathcal{O}_\beta$ is an infinite set but 0 is not an accumulation point of $\mathcal{O}_\beta$.
Class $C_4$: 0 is an accumulation point of $\mathcal{O}_\beta$ but $\mathcal{O}_\beta$ is not dense in $[0,1]$.
Class $C_5$: $\mathcal{O}_\beta$ is dense in $[0,1]$.

In [10], Schmeling proved among other things that the Class $C_5$ has full Lebesgue measure. This dense property of $\mathcal{O}_\beta$ for $\mathcal{L}$-almost all $\beta > 1$ gives us a type of hitting property, i.e., for any $x_0 \in [0,1]$ and $\mathcal{L}$-almost all $\beta > 1$,

$$\liminf_{n\to\infty} |T_\beta^n 1 - x_0| = 0.$$

Schmeling also showed that for any initial point $x \in (0,1]$, its orbit under $\beta$-transformation is dense in $[0,1]$ for $\mathcal{L}$-almost all $\beta > 1$ (see Proposition 13.1 in [10]). That is, for any $x \in (0,1]$ and $x_0 \in [0,1]$,

$$\liminf_{n\to\infty} |T_\beta^n x - x_0| = 0 \quad \text{for } \mathcal{L}\text{-a.e. } \beta > 1. \tag{1.1}$$

In this note, we consider the convergence speed in (1.1). Let $\{x_n\}_{n\geq 1} \subset [0,1]$ be a sequence of real numbers. Let $\varphi\colon \mathbb{N} \to (0,1]$ be a positive function and $\lambda(\varphi) := \limsup_{n\to\infty} \frac{\log \varphi(n)}{n}$. For any $x \in (0,1]$, define

$$E_x(\{x_n\}, \varphi) = \{\beta > 1 \colon |T_\beta^n x - x_n| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

We prove the following result.