

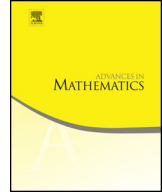


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# Diophantine analysis in beta-dynamical systems and Hausdorff dimensions <sup>☆</sup>

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## ARTICLE INFO

*Article history:*

Received 21 May 2015

Received in revised form 28

December 2015

Accepted 28 December 2015

Available online 6 January 2016

Communicated by Kenneth Falconer

*MSC:*

primary 11K55

secondary 28A80

*Keywords:*

Beta-dynamical system

Diophantine approximation

Shrinking target problem

Mass transference principle

Hausdorff dimension

## ABSTRACT

Let  $\{x_n\}_{n \geq 1} \subset [0, 1]$  be a sequence of real numbers and let  $\varphi: \mathbb{N} \rightarrow (0, 1]$  be a positive function. Using the mass transference principle established by Beresnevich and Velani [1], we prove that for any  $x \in (0, 1]$ , the Hausdorff dimension of the set

$$\{\beta > 1: |T_\beta^n x - x_n| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

satisfies a so-called 0–1 law according to  $\limsup_{n \rightarrow \infty} \frac{\log \varphi(n)}{n} = -\infty$  or not, where  $T_\beta$  is the  $\beta$ -transformation.

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## 1. Introduction

Given a real number  $\beta > 1$ , the  $\beta$ -transformation  $T_\beta: [0, 1] \rightarrow [0, 1]$  is defined by

<sup>☆</sup> This work was supported by NSFC Nos. 11225101, 11271114.

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$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor \quad \text{for all } x \in [0, 1],$$

where  $\lfloor \cdot \rfloor$  denotes the integral part of a real number. The transformation  $T_\beta$  on  $[0, 1]$  is a typical example of monotone one-dimensional expanding dynamical system. In 1957, Rényi [9] introduced this kind of map as a model for expanding a real number in a non-integer base  $\beta > 1$ . Since then, much attention has been paid to this transformation, see [2,4,6,10], etc.

For  $\beta > 1$ , the transformation  $T_\beta$  has an invariant ergodic measure  $\nu_\beta$ , which is equivalent to the Lebesgue measure  $\mathcal{L}$  on  $[0, 1]$  with the jump function

$$h_\beta(x) = \Theta(\beta) \sum_{x < T_\beta^n 1} \frac{1}{\beta^n}, \quad x \in [0, 1]$$

as its density [7] and  $\Theta(\beta)$  as the normalizing factor. This measure is the unique measure of maximal entropy [5], called the Parry measure.

Given a point  $x \in (0, 1]$ , its orbits under  $\beta$ -transformations may have completely different distributions on  $[0, 1]$  when  $\beta$  varies. For example, when  $x = 1$ , Blanchard [2] provided a classification of the parameter space  $\{\beta \in \mathbb{R} : \beta > 1\}$  according to the distributions of the orbits  $\mathcal{O}_\beta := \{T_\beta^n 1 : n \geq 1\}$ :

- Class  $C_1$ :  $\mathcal{O}_\beta$  is ultimately zero.
- Class  $C_2$ :  $\mathcal{O}_\beta$  is ultimately non-zero periodic.
- Class  $C_3$ :  $\mathcal{O}_\beta$  is an infinite set but 0 is not an accumulation point of  $\mathcal{O}_\beta$ .
- Class  $C_4$ : 0 is an accumulation point of  $\mathcal{O}_\beta$  but  $\mathcal{O}_\beta$  is not dense in  $[0, 1]$ .
- Class  $C_5$ :  $\mathcal{O}_\beta$  is dense in  $[0, 1]$ .

In [10], Schmeling proved among other things that the Class  $C_5$  has full Lebesgue measure. This dense property of  $\mathcal{O}_\beta$  for  $\mathcal{L}$ -almost all  $\beta > 1$  gives us a type of hitting property, i.e., for any  $x_0 \in [0, 1]$  and  $\mathcal{L}$ -almost all  $\beta > 1$ ,

$$\liminf_{n \rightarrow \infty} |T_\beta^n 1 - x_0| = 0.$$

Schmeling also showed that for any initial point  $x \in (0, 1]$ , its orbit under  $\beta$ -transformation is dense in  $[0, 1]$  for  $\mathcal{L}$ -almost all  $\beta > 1$  (see Proposition 13.1 in [10]). That is, for any  $x \in (0, 1]$  and  $x_0 \in [0, 1]$ ,

$$\liminf_{n \rightarrow \infty} |T_\beta^n x - x_0| = 0 \quad \text{for } \mathcal{L}\text{-a.e. } \beta > 1. \tag{1.1}$$

In this note, we consider the convergence speed in (1.1). Let  $\{x_n\}_{n \geq 1} \subset [0, 1]$  be a sequence of real numbers. Let  $\varphi : \mathbb{N} \rightarrow (0, 1]$  be a positive function and  $\lambda(\varphi) := \limsup_{n \rightarrow \infty} \frac{\log \varphi(n)}{n}$ . For any  $x \in (0, 1]$ , define

$$E_x(\{x_n\}, \varphi) = \{\beta > 1 : |T_\beta^n x - x_n| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

We prove the following result.

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