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# Gradient estimates of mean curvature equations with Neumann boundary value problems ☆

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ABSTRACT

In this paper, we use the maximum principle to get the gradient estimate for the solutions of the prescribed mean curvature equation with Neumann boundary value problem, which gives a positive answer for the question raised by Lieberman in 2013. As a consequence, we obtain the corresponding existence theorem for a class of mean curvature equations.



## 1. Introduction

Gradient estimate for the prescribed mean curvature equation has been extensively studied. The interior gradient estimate, for the minimal surface equation was obtained in the case of two variables by Finn [3]. Bombieri, De Giorgi and M. Miranda [1] obtained the estimate for high dimensional case. For the general mean curvature equation, such

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estimate had also been obtained by Ladyzhenskaya and Ural'tseva [11], Trudinger [22] and Simon [19]. All their methods were used by test function argument and a resulting Sobolev inequality. In 1983, Korevaar [8] introduced the normal variation technique and got the maximum principle proof for the interior gradient estimate on the minimal surface equation. Trudinger [23] also studied the curvature equations and got the interior gradient estimates for a class curvature equation. In 1998, Wang [25] gave a new proof for the mean curvature equation via standard Bernstein technique. The Dirichlet problem for the prescribed mean curvature equation has been studied by Jenkins–Serrin [6] and Serrin [18]. A more detailed history could be found in Gilbarg and Trudinger [5].

For the mean curvature equation with prescribed contact angle boundary value problem, Ural'tseva [24] first got the boundary gradient estimates and the corresponding existence theorem. At the same time, Simon–Spruck [20] and Gerhardt [4] also obtained existence theorem on the positive gravity case. For the $\sigma_k$-Yamabe problem on compact manifolds with boundary, Li–Zhu [12] got some existence theorem. For more general quasilinear divergence structure equation with conormal derivative boundary value problem, Lieberman [13] gave the gradient estimate. They obtained these estimates also via test function technique.

Spruck [21] used the maximum principle to obtain boundary gradient estimate in two dimensions for the positive gravity capillary problems. Korevaar [9] generalized his normal variation technique and got the gradient estimates for the positive gravity case in high dimensional case. In [14,15], Lieberman developed the maximum principle approach on the boundary gradient estimates to the quasilinear elliptic equation with oblique derivative boundary value problem, and in [16] he got the maximum principle proof for the gradient estimates on the general quasilinear elliptic equation with capillary boundary value problems.

In a recent book ([17], page 360), Lieberman posed the following question, how to get the gradient estimates for the mean curvature equation with Neumann boundary value problem. In this paper we use the technique developed by Spruck [21], Lieberman [16], Wang [25] and Jin–Li–Li [7] to get a positive answer. As a consequence, we obtain an existence theorem for a class of mean curvature equations with Neumann boundary value problem.

We first consider the boundary gradient estimates for the mean curvature equation with Neumann boundary value problem. Now let's state our main gradient estimates.

**Theorem 1.1.** *Suppose $u \in C^2(\overline{\Omega}) \bigcap C^3(\Omega)$ is a bounded solution for the following boundary value problem*

$$\mathtt{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = f(x,u) \quad in \quad \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \gamma} = \psi(x,u) \quad on \quad \partial\Omega, \tag{1.2}$$