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The cone of curves and the Cox ring of rational surfaces given by divisorial valuations [☆]

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A B S T R A C T

We consider surfaces X defined by plane divisorial valuations ν of the quotient field of the local ring R at a closed point p of the projective plane $\mathbb{P}^2$ over an arbitrary algebraically closed field k and centered at R. We prove that the regularity of the cone of curves of X is equivalent to the fact that ν is non-positive on $\mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus L)$, where L is a certain line containing p. Under these conditions, we characterize when the characteristic cone of X is closed and its Cox ring finitely generated. Equivalent conditions to the fact that ν is negative on $\mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus L) \setminus k$ are also given.

1. Introduction

In this paper we consider rational surfaces X defined by simple finite sequences of point blowing-ups starting with a blow-up at a closed point of the projective plane over

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an arbitrary algebraically closed field. We characterize those surfaces X whose cone of curves is regular and determine which ones of these are Mori dream spaces. Our surfaces are intimately related with algebraic objects, plane divisorial valuations, which will be useful in our study.

Ideas by Hensel gave rise to the concept of valuation, established in 1912 by Kürschák. This concept is an important tool in several areas of mathematics. One of the fields where valuations are very useful is that of resolution of singularities. Between 1940 and 1960, Zariski and Abhyankar applied the foundations of valuation theory to resolution of singularities of algebraic varieties [57,58,60,1,2]. It is well-known that resolution in characteristic zero was proved by Hironaka without using valuations, however many of the attempts and known results involving resolution in positive characteristic use valuation theory (see [53] as a sample).

Generally speaking, valuations are not well-known objects although some families of them have been rather studied. This is the case of valuations of quotient fields of regular two-dimensional local rings $(R, \mathfrak{m})$ centered at R (plane valuations). Zariski gave a classification for them and a refinement in terms of dual graphs can be found in [51] (see also [28]). Plane valuations are in one-to-one correspondence with simple sequences of point blowing-ups starting with the blowing-up of Spec$R$ at $\mathfrak{m}$. Here, simple means that the center of each blow-up is in the exceptional divisor produced by the previous one. Probably, and from the geometric point of view, the so-called divisorial valuations are the most interesting ones; in the case of plane valuations, they correspond with finite sequences as above and are defined by the last created exceptional divisor. Within the problem of resolution of singularities, valuations are considered as a local object and, mostly, used to treat the local uniformization problem. However, as we will see, useful global geometrical properties arise associated with certain classes of plane divisorial valuations.

Along this paper, k will denote an algebraically closed field of arbitrary characteristic, $\mathbb{P}^2 := \mathbb{P}^2_k$ the projective plane over k and our valuations will be of the quotient field of the local ring $R := \mathcal{O}_{\mathbb{P}^2,p}$, where p is a fixed point in $\mathbb{P}^2$. To fix notation, we set $(X : Y : Z)$ projective coordinates in $\mathbb{P}^2$, consider the line L with equation $Z = 0$, that will be called the line at infinity, and the point p with projective coordinates $(1 : 0 : 0)$. In addition, pick affine coordinates $x = X/Z$; $y = Y/Z$ in the chart of $\mathbb{P}^2$ given by $Z \neq 0$ and consider a divisorial valuation ν of the quotient field of the local ring R and centered at R. Set $\mathfrak{m}$ the maximal ideal of R; assume that ν is not the $\mathfrak{m}$-adic valuation (that given by $\nu(f) = s$ if and only if $f \in \mathfrak{m}^s \setminus \mathfrak{m}^{s+1}$) and L is the tangent line of ν (see Section 2.1). These conditions are assumed for all valuations we consider. A goal of this paper is either to characterize, or to provide geometrical properties of, the fact that the valuation ν is non-positive or negative on all polynomials $p(x, y)$ in the set $k[x, y] \setminus k$.

As mentioned above, some of our characterizations or properties involve a good behavior of interesting global objects as the cone of curves, the characteristic cone or the Cox ring attached to the surface that the sequence of point blowing-ups determined by ν defines.