Advances in Mathematics 290 (2016) 1138–1158

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

New effective differential Nullstellensatz[☆]

Richard Gustavson[a], Marina Kondratieva[b],
Alexey Ovchinnikov[c,d,*]

[a] CUNY Graduate Center, Department of Mathematics, 365 Fifth Avenue, New York, NY 10016, USA
[b] Moscow State University, Department of Mechanics and Mathematics, Moscow, Russia
[c] Department of Mathematics, CUNY Queens College, 65-30 Kissena Blvd, Queens, NY 11367, USA
[d] Ph.D. Program in Mathematics, CUNY Graduate Center, 365 Fifth Avenue, New York, NY 10016, USA

A R T I C L E I N F O

Article history:
Received 9 February 2015
Received in revised form 21 November 2015
Accepted 30 December 2015
Communicated by Ludmil Katzarkov

MSC:
primary 12H05
secondary 12H20, 14Q20

Keywords:
Effective differential Nullstellensatz
Differential equations
Uniform bounding

A B S T R A C T

We show new upper and lower bounds for the effective differential Nullstellensatz for differential fields of characteristic zero with several commuting derivations. Seidenberg was the first to address this problem in 1956, without giving a complete solution. In the case of one derivation, the first bound is due to Grigoriev in 1989. The first bounds in the general case appeared in 2009 in a paper by Golubitsky, Kondratieva, Szanto, and Ovchinnikov, with the upper bound expressed in terms of the Ackermann function. D'Alfonso, Jeronimo, and Solernó, using novel ideas, obtained in 2014 a new bound if restricted to the case of one derivation and constant coefficients. To obtain the bound in the present paper without this restriction, we extend this approach and use the new methods

☆ This work was partially supported by the National Science Foundation grants CCF-0952591 and DMS-1413859.

* Corresponding author at: Department of Mathematics, CUNY Queens College, 65-30 Kissena Blvd, Queens, NY 11367, USA.

E-mail addresses: rgustavson@gradcenter.cuny.edu (R. Gustavson), kondratieva@sumail.ru (M. Kondratieva), aovchinnikov@qc.cuny.edu (A. Ovchinnikov).

http://dx.doi.org/10.1016/j.aim.2015.12.021

of Freitag and León Sánchez and of Pierce, which represent a model-theoretic approach to differential algebraic geometry.

1. Introduction

It is a fundamental problem to determine whether a system $F = 0$, $F = f_1, \ldots, f_r$, of polynomial PDEs with coefficients in a differential field K is consistent, that is, it has a solution in a differential field containing K. Differential elimination [1,13] is an effective method that can answer this question, and its implementations (including Maple packages) can handle examples of moderate size if a sufficiently powerful computer is used. The differential Nullstellensatz states that the above consistency is equivalent to showing that the equation $1 = 0$ is not a differential–algebraic consequence of the system $F = 0$. Algebraically, the latter says that 1 does not belong to the differential ideal generated by F in the ring of differential polynomials.

The complexity of the effective differential Nullstellensatz is not just a central problem in the algebraic theory of partial differential equations but is also a key to understanding the complexity of differential elimination. It is often the case that this leads to substantial improvements in algorithms. Let $F = 0$ be a system of polynomial PDEs in n differential indeterminates (dependent variables) and m commuting derivation operators $\partial_1, \ldots, \partial_m$ (that is, with m independent variables), of total order h and degree d, with coefficients in a differential field K of characteristic zero. For every non-negative integer b, let $F^{(b)} = 0$ be the set of differential equations obtained from the system $F = 0$ by differentiating each equation in it b times with respect to any combination of $\partial_1, \ldots, \partial_m$. An upper bound for the effective differential Nullstellensatz is a numerical function $b(m, n, h, d)$ such that, for all such F, the system $F = 0$ is inconsistent if and only if the system of polynomial equations in $F^{(b(m,n,h,d))}$ is inconsistent. By the usual Hilbert's Nullstellensatz, the latter is equivalent to

$$1 \in \left(F^{(b(m,n,h,d))}\right).$$

For example, in the system of polynomial PDEs

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 1 \end{cases} \tag{1.1}$$

$\partial_1 = \partial/\partial_x$, $\partial_2 = \partial/\partial_y$, and so $m = 2$, the differential indeterminates are u and v, and so $n = 2$, the maximal total order of derivatives is $h = 2$, and the maximal total degree is $d = 2$. The corresponding system of polynomial equations is