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Chen ranks and resonance

Daniel C. Cohen [a,*,1], Henry K. Schenck [b,2]

[a] *Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, United States*

[b] *Department of Mathematics, University of Illinois Urbana-Champaign, Urbana, IL 61801, United States*

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The Chen groups of a group G are the lower central series quotients of the maximal metabelian quotient of G. Under certain conditions, we relate the ranks of the Chen groups to the first resonance variety of G, a jump locus for the cohomology of G. In the case where G is the fundamental group of the complement of a complex hyperplane arrangement, our results positively resolve Suciu's Chen ranks conjecture. We obtain explicit formulas for the Chen ranks of a number of groups of broad interest, including pure Artin groups associated to Coxeter groups, and the group of basis-conjugating automorphisms of a finitely generated free group.

* Corresponding author.
E-mail addresses: cohen@math.lsu.edu (D.C. Cohen), schenck@math.uiuc.edu (H.K. Schenck).
URLs: http://www.math.lsu.edu/~cohen (D.C. Cohen), http://www.math.uiuc.edu/~schenck (H.K. Schenck).

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1. Introduction

Let G be a group, with commutator subgroup $G' = [G, G]$, and second commutator subgroup $G'' = [G', G']$. The Chen groups of G are the lower central series quotients $\mathrm{gr}_k(G/G'')$ of G/G''. These groups were introduced by K.T. Chen in [4], so as to provide accessible approximations of the lower central series quotients of a link group. For example, if $G = F_n$ is the free group of rank n (the fundamental group of the n-component unlink), the Chen groups are free abelian, and their ranks, $\theta_k(G) = \mathrm{rank}\,\mathrm{gr}_k(G/G'')$, are given by

$$\theta_k(F_n) = (k-1) \cdot \binom{k+n-2}{k}, \quad \text{for } k \geq 2. \tag{1.1}$$

While apparently weaker invariants than the lower central series quotients of G itself, the Chen groups sometimes yield more subtle information. For instance, if $G = P_n$ is the Artin pure braid group, the ranks of the Chen groups distinguish G from a direct product of free groups, while the ranks of the lower central series quotients fail to do so, see [8]. In this paper, we study the Chen ranks $\theta_k(G)$ for a class of groups which includes all arrangement groups (fundamental groups of complements of complex hyperplane arrangements), and potentially fundamental groups of more general smooth quasi-projective varieties. We relate these Chen ranks to the first resonance variety of the cohomology ring of G. For arrangement groups, our results positively resolve Suciu's Chen ranks conjecture, stated in [37].

Let $A = \bigoplus_{k=0}^{\ell} A^k$ be a finite-dimensional, graded, graded-commutative, connected algebra over an algebraically closed field $\Bbbk$ of characteristic 0. For each $a \in A^1$, we have $a^2 = 0$, so right-multiplication by a defines a cochain complex

$$(A, a)\colon \quad 0 \longrightarrow A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots \xrightarrow{a} A^k \xrightarrow{a} \cdots . \tag{1.2}$$

In the context of arrangements, with A the cohomology ring of the complement, the complex (A, a) was introduced by Aomoto [1], and subsequently used by Esnault–Schechtman–Viehweg [15] and Schechtman–Terao–Varchenko [35] in the study of local system cohomology. In this context, if $a \in A^1$ is generic, the cohomology of (A, a) vanishes, except possibly in the top dimension, see Yuzvinsky [39].

In general, the resonance varieties of A, or of G in the case where $A = H^*(G; \Bbbk)$, are the cohomology jump loci of the complex (A, a),

$$\mathcal{R}^k_d(A) = \{a \in A^1 \mid \dim H^k(A, a) \geq d\},$$

homogeneous algebraic subvarieties of A^1. These varieties, introduced by Falk [16] in the context of arrangements, are isomorphism-type invariants of the algebra A. They have been the subject of considerable recent interest in a variety of areas, see, for instance,