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Hereditary C*-subalgebra lattices

Charles A. Akemann [a,*], Tristan Bice [b,1]

[a] University of California, Santa Barbara, United States
[b] Federal University of Bahia, Salvador, Brazil

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ABSTRACT

We investigate the connections between order and algebra in the hereditary C*-subalgebra lattice $\mathcal{H}(A)$ and *-annihilator ortholattice $\mathscr{P}(A)^\perp$. In particular, we characterize $\vee$-distributive elements of $\mathcal{H}(A)$ as ideals, answering a 25 year old question, allowing the quantale structure of $\mathcal{H}(A)$ to be completely determined from its lattice structure. We also show that $\mathscr{P}(A)^\perp$ is separative, allowing for C*-algebra type decompositions which are completely consistent with the original von Neumann algebra type decompositions.

* Corresponding author.
E-mail addresses: akemann@math.ucsb.edu (C.A. Akemann), Tristan.Bice@gmail.com (T. Bice).
URL: http://www.tristanbice.com (T. Bice).

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1. Introduction

1.1. Motivation

Hereditary C*-subalgebras $\mathcal{H}(A)$ of a C*-algebra A have long been considered analogs of open sets. Given the fundamental role open subsets and their lattice structure play in topological spaces (as more clearly seen in the point-free topology of frames and locales), one would expect us by now to have a deep understanding of $\mathcal{H}(A)$, with numerous theorems relating algebraic properties of A to order properties of $\mathcal{H}(A)$. But on the contrary, our knowledge of $\mathcal{H}(A)$ is still quite limited, and the study of $\mathcal{H}(A)$ has remained very much on the periphery of mainstream C*-algebra research. Needless to say, we see this as a somewhat strange state of affairs.

Another perplexing trend in operator algebras is the early divergence of von Neumann algebra and C*-algebra theory. Again, one would naturally expect that, as von Neumann algebras form a nice subclass of C*-algebras, much inspiration could be drawn from looking at the von Neumann algebra theory and trying to generalize it in various ways to C*-algebras. But yet again, we rarely see this happening, particularly in modern C*-algebra research, with much of the von Neumann algebra theory dismissed long ago as either inapplicable or irrelevant to general C*-algebras.

In fact, this is no coincidence, as it is precisely this more topological, order theoretic approach that is required to generalize some of the basic von Neumann algebra theory. This can be seen in [11] and [12], and we continue in this direction in the present paper, using mainly classical theory to prove a number of new, fundamental and very general C*-algebra results regarding $\mathcal{H}(A)$, and its subset of *-annihilators $\mathscr{P}(A)^{\perp}$. We hope this might spur on further research in this largely neglected subfield of C*-algebra theory.

1.2. Outline

We give the necessary basic definitions and assumptions for the rest of the paper in Section 2, which the reader is welcome to skim over and refer back to only when unfamiliar terminology or notation appears later on. We start the paper proper with a brief note on compactness in Section 3. Following that, we examine the semicomplement structure of $\mathcal{H}(A)$ in Section 4 and obtain various characterizations of strong orthogonality in Theorem 1. Next, in Section 5, we exhibit a correspondence, in the unital case, between two of the most common objects that appear in lattices and C*-algebras, namely complements and projections. We then show in Section 6 that the superficially similar notion of a $\wedge$-pseudocomplement turns out to have a quite different algebraic characterization in $\mathcal{H}(A)$, namely as an annihilator ideal. Next, in Section 7, we show that arbitrary ideals in $\mathcal{H}(A)$ can be characterized as the $\vee$-distributive elements, answering a long-standing question from [17]. Quantales, as introduced in [37], have often been considered the appropriate non-commutative analogs of locales, and this characterization shows that the natural quantale structure on $\mathcal{H}(A)$ is, in fact, completely determined by its lattice structure.