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# The Mabuchi geometry of finite energy classes

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ABSTRACT

We introduce different Finsler metrics on the space of smooth Kähler potentials that will induce a natural geometry on various finite energy classes $\mathcal{E}_{\tilde{\chi}}(X,\omega)$. Motivated by questions raised by R. Berman, V. Guedj and Y. Rubinstein, we characterize the underlying topology of these spaces in terms of convergence in energy and give applications of our results to existence of Kähler–Einstein metrics on Fano manifolds.


## 1. Introduction and main results

Suppose $(X, \omega)$ is a compact connected Kähler manifold and $\mathcal{H}$ is the space of Kähler potentials associated to $\omega$. We have three major goals in this paper. First, in hopes of unifying the treatment of different geometries/topologies on $\mathcal{H}$ already present in the current literature, we introduce different Finsler structures on this space. As motivation we single out two well known examples. The path length metric $d_2$ associated to the $L^2$

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Riemannian structure induces the much studied Mabuchi geometry of $\mathcal{H}$ [36]. As it turns out, by introducing the analogous $L^1$ metric and the associated path length metric $d_1$, one recovers the extremely useful strong topology introduced in [12], whose study leads to many important applications in the study of weak solutions to complex Monge–Ampére equations (see [12,4,31]). In this part we point that one can treat these different structures in a unified manner, by working with certain very general Orlicz–Finsler type metrics on $\mathcal{H}$.

Second, with applications in mind, one would like to characterize the metric geometry of these Orlicz–Finsler structures using very concrete terms. Until now, the lack of tools to give a satisfactory answer to this problem presented a formidable roadblock in drawing geometric conclusions using the $L^2$ geometry of $\mathcal{H}$. With this in mind V. Guedj [30, Section 4.3] conjectured the following pluripotential theoretic characterization:

$$\frac{1}{C}\Big(\int_X (u_1-u_0)^2\omega_{u_0}^n + \int_X (u_1-u_0)^2\omega_{u_1}^n\Big) \le d_2(u_0,u_1)^2$$
$$\le C\Big(\int_X (u_1-u_0)^2\omega_{u_0}^n + \int_X (u_1-u_0)^2\omega_{u_1}^n\Big),\ u_0,u_1\in\mathcal{H},$$

for some $C > 1$. We prove this result in the setting of the more general Orlicz–Finsler structures mentioned above. As a consequence we note that convergence in these very general spaces implies convergence in capacity of the potentials in the sense of [34] and that $\sup_X u$ is always controlled by $d_2(0,u)$ for any $u \in \mathcal{H}$.

Third, we give some immediate applications of the tools developed here. Parallelling the analogous results for the Calabi metric in [22] and answering questions posed in this same paper, we connect the existence of Kähler–Einstein metrics on Fano manifolds with stability of the Kähler–Ricci flow and properness of Ding's $\mathcal{F}$-functional measured in these general geometries. We defer other applications to a future correspondence.

### 1.1. Finsler metrics on the space of Kähler potentials

Given $(X^n,\omega)$, a connected compact Kähler manifold, the space of smooth Kähler potentials $\mathcal{H}$ is the set

$$\mathcal{H} = \{u \in C^\infty(X) |\ \omega_u := \omega + i\partial\bar\partial u > 0\}.$$

Clearly, $\mathcal{H}$ is a Fréchet manifold as an open subset of $C^\infty(X)$. For $v \in \mathcal{H}$ one can identify $T_v\mathcal{H}$ with $C^\infty(X)$. Given a normalized Young weight $\chi : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ (convex, even, lower semi-continuous and satisfying the normalizing conditions $\chi(0) = 0$, $1 \in \partial\chi(1)$) one can introduce a Finsler metric on $\mathcal{H}$ associated to the Orlicz norm of $\chi$ (see Section 2.1):

$$\|\xi\|_{\chi,u} = \inf\Big\{r > 0 : \frac{1}{\mathrm{Vol}(X)}\int_X \chi\Big(\frac{\xi}{r}\Big)\omega_u^n \le \chi(1)\Big\}, \quad \xi \in T_v\mathcal{H}. \tag{1}$$