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Stable soliton resolution for exterior wave maps in all equivariance classes ☆

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ABSTRACT

In this paper we consider finite energy ℓ-equivariant wave maps from $\mathbb{R}^{1+3}_{t,x} \setminus (\mathbb{R} \times B(0,1)) \to \mathbb{S}^3$ with a Dirichlet boundary condition at $r = 1$, and for all $\ell \in \mathbb{N}$. Each such ℓ-equivariant wave map has a fixed integer-valued topological degree, and in each degree class there is a unique harmonic map, which minimizes the energy for maps of the same degree. We prove that an arbitrary ℓ-equivariant exterior wave map with finite energy scatters to the unique harmonic map in its degree class, i.e., soliton resolution. This extends the recent results of the first, second, and fourth authors on the 1-equivariant equation to higher equivariance classes, and thus completely resolves a conjecture of Bizoń, Chmaj and Maliborski, who observed this asymptotic behavior numerically. The proof relies crucially on exterior energy estimates for the free radial wave equation in dimension $d = 2\ell + 3$, which are established in the companion paper [13].

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1. Introduction

In this paper we give a complete description of the asymptotic dynamics for the ℓ-equivariant wave map equation

$$U : \mathbb{R}^{1+3}_{t,x} \backslash (\mathbb{R} \times B(0,1)) \to \mathbb{S}^3,$$

with a Dirichlet condition on the boundary of the unit ball $B(0,1) \subset \mathbb{R}^3$ and initial data of finite energy. To be precise, consider the Lagrangian

$$\mathcal{L}(U, \partial_t U) = \int\limits_{\mathbb{R}^{1+3}_{t,x} \backslash (\mathbb{R} \times B(0,1))} \frac{1}{2} \left(-|\partial_t U|_g^2 + \sum_{j=1}^{3} |\partial_x U|_g^2 \right) dt dx,$$

where g is the round metric on $\mathbb{S}^3$, and where we only consider functions for which the boundary of the unit cylinder $\mathbb{R} \times B(0,1)$ gets mapped to a fixed point on the 3-sphere, i.e., $U(t, \partial B(0,1)) = N$, where $N \in \mathbb{S}^3$ is say, the north pole. Under the usual ℓ-equivariant assumption, for $\ell \in \mathbb{N}$, the Euler–Lagrange equation associated with this Lagrangian reduces to an equation for the azimuth angle ψ measured from the north pole on $\mathbb{S}^3$, namely

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\ell(\ell+1)}{2r^2}\sin(2\psi) = 0.$$

The Dirichlet boundary condition then becomes $\psi(t,1) = 0$ for all $t \in \mathbb{R}$ and thus the Cauchy problem under consideration is,

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\ell(\ell+1)}{2r^2}\sin(2\psi) = 0, \quad r \geq 1, \\ &\psi(t,1) = 0, \quad \forall t, \\ &\psi(0,r) = \psi_0(r), \ \psi_t(0,r) = \psi_1(r), \end{aligned} \tag{1.1}$$

and solutions $\vec{\psi}(t) := (\psi(t), \psi_t(t))$ to (1.1) will be referred to as ℓ-equivariant exterior wave maps. The conserved energy for (1.1) is given by

$$\mathcal{E}_\ell(\psi, \psi_t) = \int_1^\infty \frac{1}{2} \left(\psi_t^2 + \psi_r^2 + \frac{\ell(\ell+1)\sin^2\psi}{r^2} \right) r^2 \, dr.$$

A simple analysis of the last term in the integrand above yields topological information on the wave map if we require the energy to be finite. Indeed, any $\vec{\psi}(t,r)$ with finite energy and continuous dependence on $t \in I = (t_0, t_1)$ must satisfy $\psi(t,\infty) = n\pi, \forall t \in I$, where $n \in \mathbb{Z}$. Given the fact that ψ measures the azimuth angle from the north pole, and $\psi(t,1) = 0$ for all $t \in I$, this means that the integer $|n|$ measures the winding number,