Advances in Mathematics 285 (2015) 1301–1338

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Spectral analysis of selfadjoint elliptic differential operators, Dirichlet-to-Neumann maps, and abstract Weyl functions

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A R T I C L E I N F O

Article history:
Received 11 June 2014
Received in revised form 13 July 2015
Accepted 23 August 2015
Available online 8 September 2015
Communicated by N.G. Makarov

Keywords:
Elliptic differential operator
Dirichlet-to-Neumann map
Spectral analysis
Weyl function
Boundary triple

A B S T R A C T

The spectrum of a selfadjoint second order elliptic differential operator in $L^2(\mathbb{R}^n)$ is described in terms of the limiting behavior of Dirichlet-to-Neumann maps, which arise in a multi-dimensional Glazman decomposition and correspond to an interior and an exterior boundary value problem. This leads to PDE analogs of renowned facts in spectral theory of ODEs. The main results in this paper are first derived in the more abstract context of extension theory of symmetric operators and corresponding Weyl functions, and are applied to the PDE setting afterwards.

1. Introduction

The Titchmarsh–Weyl function is an indispensable tool in direct and inverse spectral theory of ordinary differential operators and more general systems of ordinary differential equations; see the classical monographs [15,60] and [9,16,28–30,37,41,47,56,57] for a small selection of more recent contributions. For a singular second order Sturm–Liouville

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http://dx.doi.org/10.1016/j.aim.2015.08.016
0001-8708/

differential operator of the form $\mathfrak{L}_+ = -\frac{d^2}{dx^2} + q_+$ on $\mathbb{R}_+$ with a real-valued, bounded potential q_+ the Titchmarsh–Weyl function m_+ can be defined as

$$m_+(\lambda) = \frac{f'_\lambda(0)}{f_\lambda(0)}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1.1}$$

where f_λ is a square-integrable solution of $\mathfrak{L}_+ f = \lambda f$ on $\mathbb{R}_+$; cf. [60,61]. The function $m_+ : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ belongs to the class of Nevanlinna (or Riesz–Herglotz) functions and it is a celebrated fact that it reflects the complete spectral properties of the selfadjoint realizations of $\mathfrak{L}_+$ in $L^2(\mathbb{R}_+)$. E.g. the eigenvalues of the Dirichlet realization A_D are precisely those $\lambda \in \mathbb{R}$, where $\lim_{\eta \searrow 0} i\eta m_+(\lambda + i\eta) \neq 0$, the isolated eigenvalues among them coincide with the poles of m_+, and the absolutely continuous spectrum of A_D (roughly speaking) consists of all λ with the property $0 < \operatorname{Im} m_+(\lambda + i0) < +\infty$.

If $\mathfrak{L} = -\frac{d^2}{dx^2} + q$ is a singular Sturm–Liouville expression on $\mathbb{R}$ with q real-valued and bounded, it is most natural to use decomposition methods of Glazman type for the analysis of the corresponding selfadjoint operator in $L^2(\mathbb{R})$; cf. [31]. More precisely, the restriction of $\mathfrak{L}$ to $\mathbb{R}_+$ gives rise to the Titchmarsh–Weyl function m_+ in (1.1), and similarly a Titchmarsh–Weyl function m_- associated to the restriction of $\mathfrak{L}$ to $\mathbb{R}_-$ is defined. In that case usually the functions

$$m(\lambda) = -\big(m_+(\lambda) + m_-(\lambda)\big)^{-1} \quad \text{and} \quad \widetilde{m}(\lambda) = \begin{pmatrix} -m_+(\lambda) & 1 \\ 1 & m_-(\lambda)^{-1} \end{pmatrix}^{-1} \tag{1.2}$$

are employed for the description of the spectrum. Whereas the scalar function m seems to be more convenient it will in general not contain the complete spectral data, a drawback that is overcome when using the 2×2-matrix function $\widetilde{m}$. Some of these observations were already made in [39,60], similar ideas can also be found in [36,38,42] for Hamiltonian systems, and more recently in an abstract operator theoretical framework in [17,19], see also [6].

One of the main objectives of this paper is to extend the classical spectral analysis of ordinary differential operators via the Titchmarsh–Weyl functions in (1.2) to the multidimensional setting. For this consider the second order partial differential expression

$$\mathcal{L} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{a_j} \right) + a \tag{1.3}$$

with smooth, bounded coefficients $a_{jk}, a_j : \mathbb{R}^n \to \mathbb{C}$ and $a : \mathbb{R}^n \to \mathbb{R}$ bounded, and assume that $\mathcal{L}$ is formally symmetric and uniformly elliptic on $\mathbb{R}^n$. Let A be the selfadjoint operator associated to (1.3) in $L^2(\mathbb{R}^n)$. Our main goal is to describe the spectral data of A, that is, isolated and embedded eigenvalues, continuous, absolutely continuous and singular continuous spectral points, in terms of the limiting behavior of appropriate multidimensional counterparts of the functions in (1.2). Note first that the multidimensional analogue of the Titchmarsh–Weyl function (1.1) is the Dirichlet-to-Neumann map, and