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Applications of the Funk–Hecke theorem to smoothing and trace estimates

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For a wide class of Kato-smoothing estimates with radial weights, the Funk–Hecke theorem is used to generate a new expression for the optimal constant in terms of the Fourier transform of the weight, from which several applications are given. For example, we are able to easily establish a unified theorem, assuming natural power-like asymptotic estimates for the Fourier transform of the weight, from which many well-studied smoothing estimates immediately follow, as well as sharpness of the decay and smoothness exponents. Furthermore, observing that the weight has an everywhere positive Fourier transform in many well-studied cases, our approach allows sharper information regarding the optimal constant and extremisers, substantially extending earlier work of Simon. These observations are very closely related to the Mizohata–Takeuchi conjecture regarding the equivalence of weighted L^2 bounds for the Fourier extension operator on the sphere and the uniform boundedness of the X-ray transform of the weight. For radial weights, this has been independently established by Barceló–Ruiz–Vega and Carbery–Soria; we provide a short alternative proof in three and higher dimensions of this equivalence when the Fourier transform of the weight is positive, with the optimal relationship between constants. Finally, our approach works for the closely connected trace theorems on the sphere where

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analogous results are given, including the optimal constant and characterisation of extremisers for the inhomogeneous $H^s(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ trace theorem.

1. Introduction

The Funk–Hecke theorem is a fundamental result from the theory of spherical harmonics which has found a number of applications beyond classical harmonic analysis, including convex geometry and partial differential equations, and in some cases has facilitated strikingly short proofs of important results. Very recently it was used in [7] to show that for the solution of the free Schrödinger equation, the smoothing estimate

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |D^a e^{it\Delta} f(x)|^2 \frac{\mathrm{d}x\mathrm{d}t}{|x|^{2(1-a)}} \leq \mathbf{C}(a,d) \|f\|^2_{L^2(\mathbb{R}^d)}, \tag{1.1}$$

holds for $a \in (1 - \frac{d}{2}, \frac{1}{2})$ and $d \geq 2$, with optimal constant

$$\mathbf{C}(a,d) = \pi 2^{2a-1} \frac{\Gamma(1-2a)\Gamma(\frac{d}{2}+a-1)}{\Gamma(1-a)^2\Gamma(\frac{d}{2}-a)}$$

which is attained if and only if f is radially symmetric. Here, we use D^a to denote the operator given by $\widehat{D^a f} = |\cdot|^a \widehat{f}$. Using different arguments, such sharp estimates in the case $a = 0$ and $d \geq 3$ were first established by Simon [16], and Watanabe [24] extended these to the full range of a and d.

It is well-known that there is a direct connection between such smoothing estimates and trace theorems for the sphere, and building on this, it was shown in [5] that the Funk–Hecke theorem yields the homogeneous trace estimates

$$\|f|_{\mathbb{S}^{d-1}}\|^2_{L^2(\mathbb{S}^{d-1})} \leq \frac{1}{\pi}\mathbf{C}(1-s,d)\|f\|^2_{\dot{H}^s(\mathbb{R}^d)} \tag{1.2}$$

for $s \in (\frac{1}{2}, \frac{d}{2})$ and $d \geq 2$, where $\dot{H}^s(\mathbb{R}^d)$ denotes the homogeneous Sobolev space of order s. The optimality of the constant was shown in [15] using the fact that (1.1) formally implies (1.2); the alternative and direct approach in [5] provided additional information, including a characterisation of the extremisers.

Underlying the proofs of (1.1) and (1.2) in [5] and [7] is a stronger spectral decomposition of T^*T in terms of spherical harmonics, where T is some linear operator arising from either the solution operator $f \mapsto e^{it\Delta}f$ or the trace operator $f \mapsto f|_{\mathbb{S}^{d-1}}$. These stronger results rely heavily on the homogeneity of the spatial weight $|x|^{2(a-1)}$ in (1.1) or the homogeneous Sobolev norm on the right-hand side of (1.2). Nevertheless, in this paper we show that one can use the Funk–Hecke theorem to give a new expression for