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A complete solution of Markov's problem on connected group topologies

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A B S T R A C T

Every proper closed subgroup of a connected Hausdorff group must have index at least $\mathfrak{c}$, the cardinality of the continuum. 70 years ago Markov conjectured that a group G can be equipped with a connected Hausdorff group topology provided that every subgroup of G which is closed in *all* Hausdorff group topologies on G has index at least $\mathfrak{c}$. Counter-examples in the non-abelian case were provided 25 years ago by Pestov and Remus, yet the problem whether Markov's Conjecture holds for abelian groups G remained open. We resolve this problem in the positive.

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As usual, $\mathbb{Z}$ denotes the group of integers, $\mathbb{Z}(n)$ denotes the cyclic group of order n, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{P}$ denotes the set of all prime numbers, $|X|$ denotes the cardinality of a set X, $\mathfrak{c}$ denotes the cardinality of the continuum and ω denotes the cardinality of $\mathbb{N}$.

Let G be an abelian group. For a cardinal σ, we use $G^{(\sigma)}$ to denote the direct sum of σ many copies of the group G. For $m \in \mathbb{N}$, we let

$$mG = \{mg : g \in G\} \quad \text{and} \quad G[m] = \{g \in G : mg = 0\},$$

where 0 is the zero element of G. A group G is *bounded* (or has *finite exponent*) if $mG = \{0\}$ for some integer $m \geq 1$; otherwise, G is said to be *unbounded*. We denote by

$$t(G) = \bigcup_{m \in \mathbb{N}} G[m] \tag{1}$$

the *torsion subgroup* of G. The group G is *torsion* if $t(G) = G$.

As usual, we write $G \cong H$ when groups G and H are isomorphic.

We refer the reader to [6,14,15,18,19] for standard notions related to algebra, topology and topological groups.

All topological groups and all group topologies are assumed to be Hausdorff.

1. Markov's problem for abelian groups

Markov [21,22] says that a subset X of a group G is *unconditionally closed* in G if X is closed in every Hausdorff group topology on G.

Every proper closed subgroup of a connected group must have index at least $\mathfrak{c}$.[1] Therefore, if a group admits a connected group topology, then all its proper unconditionally closed subgroups necessarily have index at least $\mathfrak{c}$; see [23]. Markov [23, Problem 5, p. 271] asked if the converse is also true.

Problem 1.1. If all proper unconditionally closed subgroups of a group G have index at least $\mathfrak{c}$, does then G admit a connected group topology?

Definition 1.2. For brevity, a group satisfying Markov's condition, namely having all proper unconditionally closed subgroups of index at least $\mathfrak{c}$, shall be called an *M-group* (the abbreviation for *Markov group*).

Adopting this terminology, Markov's Problem 1.1 reads: *Does every M-group admit a connected group topology?*

[1] Indeed, if H is a proper closed subgroup of a connected group G, the quotient space G/H is non-trivial, connected and completely regular. Since completely regular spaces of size less than $\mathfrak{c}$ are disconnected, this shows that $|G/H| \geq \mathfrak{c}$.