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Unique continuation from infinity for linear waves

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We prove various uniqueness results from null infinity, for linear waves on asymptotically flat space–times. Assuming vanishing of the solution to infinite order on suitable parts of future and past null infinities, we derive that the solution must vanish in an open set in the interior. We find that the parts of infinity where we must impose a vanishing condition depend strongly on the background geometry. In particular, for backgrounds with positive mass (such as Schwarzschild or Kerr), the required assumptions are much weaker than the ones in the Minkowski space–time. The results are nearly optimal in many respects. They can be considered analogues of uniqueness from infinity results for second order elliptic operators. This work is partly motivated by questions in general relativity.

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1. Introduction

We prove unique continuation results from infinity for wave equations over asymptotically flat backgrounds. In particular, consider solutions ϕ of a linear wave equation

$$L_g\phi := \Box_g\phi + a^\alpha\partial_\alpha\phi + V\phi = 0 \tag{1.1}$$

over a Lorentzian manifold (M, g), with $\Box_g$ being the Laplace–Beltrami operator for g. We show that if the solution vanishes (in a suitable sense) at parts of future and past null infinities $\mathcal{I}^+, \mathcal{I}^- \subset \partial M$, then the solution ϕ must vanish on an open domain inside (M, g).

One motivation for this paper comes from older and newer studies in general relativity regarding the possibility of periodic-in-time solutions of the Einstein equations; see [3] and discussion therein. This has been considered both for vacuum space–times, [33–35], and for gravity coupled with matter fields, [7,8]. In most settings, the problem reduces to whether an asymptotic symmetry of the metric at null infinities $\mathcal{I}^+, \mathcal{I}^-$ can be extended to a genuine symmetry of the space–time. In view of the techniques developed in [2], the theorems obtained here are developed to apply to such a nonlinear setting. Accordingly in [3], we have employed the results of this paper to rule out the existence of genuinely time-periodic solutions to the Einstein vacuum equations, at least near infinity in the asymptotically flat setting. In doing so, we have to overcome certain difficulties which are specific to the problem at hand, and do not appear to have been treated in the literature:

As we discuss below, the Carleman estimates on which our method relies necessitate the construction of a pseudo-convex function. This in turn requires an understanding of the geometry of null geodesics near spatial infinity. Our method here rests on a novel connection between pseudo-convexity and the ADM mass of the underlying space–time. The effect of the mass is manifested in results that are much stronger in the positive-mass setting compared to the (zero-mass) perturbed Minkowski space–times.

In order to convert our constructed pseudo-convex function into unique continuation results, one faces difficulties from the fact that we seek to perform continuation from suitable *parts* of null infinities. To overcome these, we apply novel conformal "inversions" that transform the problem at hand to that of unique continuation of waves from a double null cone in a *singular* space–time. This singularity turns out to either be innocuous (in the zero-mass setting) or to actually *help* us in the positive mass settings; we solve the transformed problem in a geometrically robust manner via a new family of general Carleman estimates.