Advances in Mathematics 286 (2016) 703–721

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Cone-volume measure of general centered convex bodies

Károly J. Böröczky [a,1], Martin Henk [b,*]

[a] *Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reltanoda u. 13-15, H-1053 Budapest, Hungary*
[b] *Technische Universität Berlin, Institut für Mathematik, Sekr. Ma 4-1, Straße des 17 Juni 136, D-10623 Berlin, Germany*

A R T I C L E   I N F O

*Article history:*
Received 16 February 2015
Received in revised form 21 September 2015
Accepted 29 September 2015
Available online 8 October 2015
Communicated by Erwin Lutwak

Dedicated to the memory of our friend Ulrich Betke

*MSC:*
52A40
52A20

*Keywords:*
Cone-volume measure
Subspace concentration condition
U-functional
Centro-affine inequalities
log-Minkowski problem
$L_p$-Minkowski problem
Centroid
Polytope

A B S T R A C T

We show that the cone-volume measure of a convex body with centroid at the origin satisfies the subspace concentration condition. This extends former results obtained in the discrete as well as in the symmetric case and implies, among others, a conjectured best possible inequality for the U-functional of a convex body.



* Corresponding author.
*E-mail addresses:* carlos@renyi.hu (K.J. Böröczky), henk@math.tu-berlin.de (M. Henk).

[1] The research of the first author is supported by OTKA 109789.

http://dx.doi.org/10.1016/j.aim.2015.09.021
0001-8708/

## 1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ having non-empty interiors, i.e., $K \in \mathcal{K}^n$ is a convex compact subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$ with $\mathrm{int}\,(K) \neq \emptyset$. As usual, we denote by $\langle\cdot,\cdot\rangle$ the inner product on $\mathbb{R}^n \times \mathbb{R}^n$ with associated Euclidean norm $\|\cdot\|$, and $S^{n-1} \subset \mathbb{R}^n$ denotes the $(n-1)$-dimensional unit sphere, i.e., $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

For $K \in \mathcal{K}^n$ we write $S_K(\cdot)$ and $h_K(\cdot)$ to denote its surface area measure and support function, respectively, and $\nu_K$ to denote the Gauß map assigning the outer unit normal $\nu_K(x)$ to an $x \in \partial_* K$, where $\partial_* K$ consists of all points in the boundary $\partial K$ of $K$ having a unique outer normal vector. If the origin $o$ lies in $K \in \mathcal{K}^n$, the *cone-volume measure of* $K$ on $S^{n-1}$ is given by

$$\mathrm{V}_K(\omega) = \int\limits_{\omega} \frac{h_K(u)}{n}\,\mathrm{d}S_K(u) = \int\limits_{\nu_K^{-1}(\omega)} \frac{\langle x, \nu_K(x)\rangle}{n}\,\mathrm{d}\mathcal{H}_{n-1}(x), \tag{1.1}$$

where $\omega \subseteq S^{n-1}$ is a Borel set and, in general, $\mathcal{H}_k(x)$ denotes the $k$-dimensional Hausdorff measure. Instead of $\mathcal{H}_n(\cdot)$, we also write $\mathrm{V}(\cdot)$ for the $n$-dimensional volume.

The name cone-volume measure stems from the fact that if $K$ is a polytope with facets $F_1, \ldots, F_m$ and corresponding outer unit normals $u_1, \ldots, u_m$, then

$$\mathrm{V}_K(\omega) = \sum_{i=1}^{m} \mathrm{V}([o, F_i])\delta_{u_i}(\omega). \tag{1.2}$$

Here $\delta_{u_i}$ is the Dirac delta measure on $S^{n-1}$ concentrated at $u_i$, and for $x_1, \ldots, x_m \in \mathbb{R}^n$ and subsets $S_1, \ldots, S_l \subseteq \mathbb{R}^n$ we denote the convex hull of the set $\{x_1, \ldots, x_m, S_1, \ldots, S_l\}$ by $[x_1, \ldots, x_m, S_1, \ldots, S_l]$. With this notation $[o, F_i]$ is the cone with apex $o$ and basis $F_i$.

In recent years, cone-volume measures have appeared and were studied in various contexts, see, e.g., F. Barthe, O. Guedon, S. Mendelson and A. Naor [6], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [10,11], M. Gromov and V.D. Milman [18], M. Ludwig [28], M. Ludwig and M. Reitzner [29], E. Lutwak, D. Yang and G. Zhang [32], A. Naor [34], A. Naor and D. Romik [35], G. Paouris and E. Werner [36], A. Stancu [42], G. Zhu [45,46], K.J. Böröczky and P. Hegedűs [8].

In particular, cone-volume measures are the subject of the *logarithmic Minkowski problem*, which is the particular interesting limiting case $p = 0$ of the general $L_p$-Minkowski problem – one of the central problems in convex geometric analysis. It is the task:

*Find necessary and sufficient conditions for a finite Borel measure $\mu$ on $S^{n-1}$ to be the cone-volume measure $\mathrm{V}_K$ of $K \in \mathcal{K}^n$ with $o$ in its interior.*

In the recent paper [11], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang solved the logarithmic Minkowski problem in the even case, i.e., they characterized the cone-volume