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Milnor–Witt K-groups of local rings ☆

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ABSTRACT

We introduce Milnor–Witt K-groups of local rings and show that the nth Milnor–Witt K-group of a local ring R which contains an infinite field of characteristic not 2 is the pull-back of the nth power of the fundamental ideal in the Witt ring of R and the nth Milnor K-group of R over the nth Milnor K-group of R modulo 2. This generalizes the work of Morel–Hopkins on Milnor–Witt K-groups of a field.

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0. Introduction

The Milnor–Witt K-theory of a field R, denoted $\mathrm{K}_*^{MW}(R) = \bigoplus_{n\in\mathbb{Z}} \mathrm{K}_n^{MW}(R)$, arises as an object of fundamental interest in motivic homotopy theory; namely, as the "0-line" part of the stable homotopy ring of the motivic sphere spectrum over R, see Morel [22]. Beginning from an initial presentation discovered in collaboration with Hopkins, Morel [21] showed that, for a field R, the group $\mathrm{K}_n^{MW}(R)$ is the pull-back of the diagram

$$\begin{array}{ccc} & & \mathrm{K}_n^M(R) \\ & & \downarrow{\scriptstyle e_n} \\ \mathrm{I}^n(R) & \longrightarrow & \mathrm{I}^n(R)/\,\mathrm{I}^{n+1}(R)\,, \end{array} \tag{1}$$

where $\mathrm{K}_n^M(R)$ denotes the nth Milnor K-group of R, the symbol $\mathrm{I}^n(R)$ the nth power of fundamental ideal $\mathrm{I}(R)$ in the Witt ring $\mathrm{W}(R)$, and e_n maps the symbol $\ell(a_1)\cdot \ldots \cdot \ell(a_n)$ to the class of the Pfister form $\ll a_1, \ldots, a_n \gg$. (Here $\mathrm{I}^n(R) = \mathrm{W}(R)$ and $\mathrm{K}_n^M(R) = 0$ is understood for $n < 0$.) In this form, Milnor–Witt K-groups of fields already appeared implicitly in earlier work of Barge and Morel [7], who used them to introduce oriented Chow groups of an algebraic variety, a theory later elaborated in the work of Fasel [11,12]. This so-called Chow–Witt theory was used by Barge and Morel [7,8] to construct an Euler class invariant for algebraic oriented vector bundles which, in analogy with its topological counterpart, is the primary obstruction to the existence of a nowhere-vanishing section, see Morel [23, Chap. 8]. This circle of ideas has since then been greatly elaborated upon, see for instance the article [3] by Asok and Fasel for some very recent developments.

A further area where Milnor–Witt K-groups of a field show up is the theory of framed motives, see for instance the recent work of Neshitov [24].

We introduce in this work Milnor–Witt K-groups $\mathrm{K}_*^{MW}(R)$ of a local ring R. Our definition is the naive generalization of the Morel–Hopkins presentation given in Morel's book [23, Def. 3.1], or Morel [21, Def. 5.1], *i.e.* $\mathrm{K}_*^{MW}(R)$ is a $\mathbb{Z}$-graded $\mathbb{Z}$-algebra generated by an element $\hat{\eta}$ in degree -1 and elements $\{a\}$ (a is a unit in R) in degree 1 modulo four relations, see Definition 5.1. This definition has also been considered in the recent work of Schlichting [29], where (extending earlier work of Barge and Morel [8] and Hutchinson and Tao [14] over fields) it is shown that Milnor–Witt K-groups arise as obstructions to integral homology stability for special linear groups over local rings with infinite residue fields. This in turn is used then by Schlichting [29, §6] to extend results of Morel on the vanishing of the Euler class for oriented vector bundles over affine schemes.

Our main result about these groups is the following, see Theorem 5.4.