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Surveying points in the complex projective plane

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ABSTRACT

We classify SIC-POVMs of rank one in $\mathbb{CP}^2$, or equivalently sets of nine equally-spaced points in $\mathbb{CP}^2$, without the assumption of group covariance. If two points are fixed, the remaining seven must lie on a pinched torus that a standard moment mapping projects to a circle in $\mathbb{R}^3$. We use this approach to prove that any SIC set in $\mathbb{CP}^2$ is isometric to a known solution, given by nine points lying in triples on the equators of the three 2-spheres each defined by the vanishing of one homogeneous coordinate. We set up a system of equations to describe hexagons in $\mathbb{CP}^2$ with the property that any two vertices are related by a cross ratio (transition probability) of 1/4. We then symmetrize the equations, factor out by the known solutions, and compute a Gröbner basis to show that no SIC sets remain. We do find new configurations of nine points in which 27 of the 36 pairs of vertices of the configuration are equally spaced.

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0. Introduction

A symmetric, informationally complete, positive-operator valued measure or SIC-POVM on the Hermitian vector space $\mathbb{C}^n$ is a set $\{P_j\}$ of n^2 rank-one projection operators such that

$$\frac{1}{n}\sum_{i=1}^{n^2} P_i = I,$$

and

$$\mathrm{tr}(P_j P_k) = \frac{1}{n+1}(n\delta_{jk} + 1)$$

for all j, k. Such objects attracted wide attention following conjectures about their existence made by Zauner [43] in 1999 and Renes et al. [32] in 2004, and since then have been investigated by a large number of authors, along with higher rank versions and the allied concept of mutually unbiased basis. See, for example, [2,4,3,11,13,17,18,21,26,33,42,44], and references cited therein.

SIC-POVMs arise in the theory of quantum measurement (see Davies [12] and Holevo [22] for the significance of general POVMs), and are of great interest in connection with their potential applications to quantum tomography. The idea is the following. Suppose that one has a large number of independent identical copies of a quantum system (say, a large molecule), the state (or 'structure') of which is unknown and needs to be determined. A SIC-POVM can be thought of as a kind of symmetrically oriented machine that can be used to make a single tomographic measurement on each independent copy of the molecule, with the property that once the results of the various measurements have been gathered for a sufficiently large number of molecules, the state of the molecule can be efficiently determined to a high degree of accuracy. The 'symmetric orientation' is not with respect to ordinary three-dimensional physical space (as in the classical tomography of medical imaging), but rather with respect to the space of pure quantum states.

Since each element P_i of a SIC-POVM is a matrix of rank one and trace unity, it determines a point in complex projective space $\mathbb{CP}^{n-1}$. It is well known that a SIC-POVM can then be defined as a configuration of n^2 points in $\mathbb{CP}^{n-1}$ that are mutually equidistant under the standard Kähler metric [29,39]. This is the definition that we shall adopt in Section 3, and the distance is determined by Lemma 3.6. Such a set of points is often called a 'SIC', but we favour the expression 'SIC set'.

The existence of such configurations (for example, nine equidistant points in $\mathbb{CP}^2$, or sixteen equidistant points in $\mathbb{CP}^3$) is counterintuitive to our everyday way of thinking in which a regular simplex in $\mathbb{R}^n$ has $n+1$ vertices (but see [19]). It has been conjectured that $\mathbb{CP}^{n-1}$ possesses such a configuration for every n [32,43]. There is evidence for this for n up to at least 67, and various explicit solutions have been found in lower dimensions. Most of the known SIC sets in higher dimensions are constructed as orbits