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Effective uniform bounding in partial differential fields ☆

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ABSTRACT

Motivated by the effective bounds found in [12] for ordinary differential equations, we prove an effective version of uniform bounding for fields with several commuting derivations. More precisely, we provide an upper bound for the size of finite solution sets of partial differential polynomial equations in terms of data explicitly given in the equations and independent of parameters. Our methods also produce an upper bound for the degree of the Zariski closure of solution sets, whether they are finite or not.

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1. Introduction

Suppose we are given a system of partial differential polynomial equations over $\mathbb{Q}$,

$$\begin{aligned} p_1(x,y) &= 0 \\ p_2(x,y) &= 0 \\ &\vdots \\ p_r(x,y) &= 0 \end{aligned}$$

so that for some specific values of $y = (y_1, \dots, y_s)$ in some differentially closed field $(K, \Delta = \{\delta_1, \dots, \delta_m\})$ of characteristic zero with commuting derivations, the number of solutions (in the variables $x = (x_1, \dots, x_n)$) in K^n is finite. Can one bound the number of solutions in terms of the basic invariants of the differential polynomials p_i without any reference to the selected values of y? More generally, without assuming finiteness of the solution set: Can one bound the degree of its Zariski closure? In this paper we will answer these questions affirmatively, and give bounds which depend only on the order, degree, and number of variables in the differential polynomials (and the number of derivations).

Besides being a problem of foundational interest, this problem is intimately connected to the effective differential Nullstellensatz, and is also at the heart of applications of differential algebra to the so-called special points conjectures in number theory. We will give more details of these connections after discussing some of the history and difficulties of this problem.

Remark 1.1. For the model theorist, the existence of this type of effective bounds implies, among other things, that the theory $\mathrm{DCF}_{0,m}$ has *uniform bounding* which seems to be a new result in the case of fields with several commuting derivations (the effective bounds found in [12] imply uniform bounding for the ordinary case, for a noneffective proof see [17]). Consequently, since differentially closed fields are stable and eliminate imaginaries, a result of [29] implies that $\mathrm{DCF}_{0,m}$ has NFCP (the non-finite cover property).

We now remark on the difficulties that arise (in the case of several commuting derivations) while trying to find effective bounds. The case of a single derivation was considered in [12]; let us briefly describe the methods of that paper. Assume $\Delta = \{\delta\}$, and let us consider the case of first-order differential equations. In this case, the problem can be restated as follows: Are there effective upper bounds for the size of finite sets of the form $Z = \{v \in V : (v, \delta(v)) \in W\}$ where V and W are algebraic varieties?

Fact 1.2. *(In the ordinary case.) Let V and W be closed subvarieties of K^n and K^{2n}, respectively. If $Z = \{v \in V : (v, \delta(v)) \in W\}$ is finite, then*

$$|Z| \leq (\deg V)^{2^{\dim V}} (\deg W)^{2^{\dim V} - 1}.$$