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Affine symmetries of orbit polytopes

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ABSTRACT

An orbit polytope is the convex hull of an orbit under a finite group $G \leqslant \mathrm{GL}(d, \mathbb{R})$. We develop a general theory of possible affine symmetry groups of orbit polytopes. For every group, we define an open and dense set of generic points such that the orbit polytopes of generic points have conjugated affine symmetry groups. We prove that the symmetry group of a generic orbit polytope is again G if G is itself the affine symmetry group of some orbit polytope, or if G is absolutely irreducible. On the other hand, we describe some general cases where the affine symmetry group grows.

We apply our theory to representation polytopes (the convex hull of a finite matrix group) and show that their affine symmetries can be computed effectively from a certain character. We use this to construct counterexamples to a conjecture of Baumeister et al. on permutation polytopes (Baumeister et al., 2009 [4, Conjecture 5.4]).

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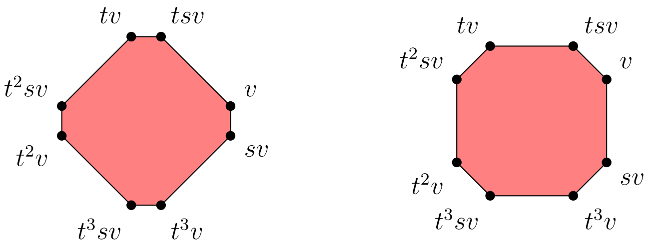

Fig. 1. Two typical orbit polytopes of $D_4 = \langle t, s \rangle$, the group of the square. Both have no additional affine symmetries.

1. Introduction

Let $G \leqslant \mathrm{GL}(d, \mathbb{R})$ be a finite group. An **orbit polytope** of G is defined as the convex hull of the orbit Gv of some point $v \in \mathbb{R}^d$. We denote it by

$$P(G, v) = \mathrm{conv}\{gv \mid g \in G\}.$$

Orbit polytopes have been studied by a number of authors [1,2,13,35,37], especially orbit polytopes of finite reflection groups, which are often called generalized permutahedra, or simply permutahedra [6,17,18,26,42]. Let us mention here that the classical *Wythoff construction* [8–10] basically consists in taking orbits under a reflection group to construct polytopes or tesselations of a sphere. In particular, Coxeter [8] has shown that several uniform polytopes can be obtained as orbit polytopes of (finite) reflection groups by choosing a suitable starting point v (see also [34]). In the language of Sanyal, Sottile and Sturmfels [38], orbit polytopes are polytopal *orbitopes*. (An orbitope is the convex hull of an orbit of a compact group, not necessarily finite.)

In this paper we study the affine symmetry groups of orbit polytopes. An **affine symmetry** of a polytope $P \subset \mathbb{R}^d$ is a bijection of P which is the restriction of an affine map $\mathbb{R}^d \to \mathbb{R}^d$. We write $\mathrm{AGL}(P)$ for the affine symmetry group of a polytope P.

Clearly, the affine symmetry group of an orbit polytope $P(G, v)$ always contains the symmetries induced by G. Depending on the group and on the point v, there may be additional symmetries or not. In particular, certain symmetry groups imply additional symmetries for all orbit polytopes. In this paper we develop a general theory to explain this phenomenon. We begin by looking at some very simple examples.

1.1. Illustrating examples

Let $G = \langle t, s \rangle \cong D_4$, the dihedral group[2] of order 8. Here t denotes a counterclockwise rotation by a right angle, and s a reflection (in the plane). Fig. 1 shows two "generic" orbit polytopes. Their affine symmetry group is only the group G itself. In contrast, the

[2] In this paper, we follow the convention of geometers and write D_n for the group of the n-gon with $2n$ elements. Most group theorists write D_{2n} instead.