Advances in Mathematics 288 (2016) 426–463

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Suffridge's convolution theorem for polynomials and entire functions having only real zeros

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ARTICLE INFO

Article history:
Received 16 January 2013
Received in revised form 12 August 2015
Accepted 23 October 2015
Available online xxxx
Communicated by George E. Andrews

MSC:
30C10
30C15
26C10
30D15
05A99

Keywords:
Suffridge polynomials
Pólya–Schur multiplier sequences
q-Binomials
Log-concave sequences
Newton's inequalities

ABSTRACT

We present a q-extension of Pólya's and Schur's characterization of multiplier sequences and a q-extension and a converse of Newton's inequalities.

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[1] The research that led to this paper was conducted while I had a post-doc position at the department of mathematics at the University of Würzburg. I am very grateful to Stephan Ruscheweyh and all members of Lehrstuhl IV for the productive and friendly atmosphere there. I also would like to thank the referee for pointing out several interesting references and for inventing the term 'q-function theory'.

http://dx.doi.org/10.1016/j.aim.2015.10.016
0001-8708/

1. Introduction

In [24] Rota states: "Grace's theorem is an instance of what might be called a sturdy theorem. For almost one hundred years it has resisted all attempts at generalization. Almost all known results about the distribution of zeros of polynomials in the complex plane are corollaries of Grace's theorem."

The following equivalent formulation of Grace's theorem is due to Szegö.

Theorem 1. *(See Grace [11], Szegö [34].) Let*

$$F(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \quad \text{and} \quad G(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

be polynomials of degree $n \in \mathbb{N}$ and suppose $K \subset \mathbb{C}$ is an open or closed disk or half-plane, or the open or closed exterior of a disk, that contains all zeros of F. If $G(0) \neq 0$, then each zero γ of

$$F *_{GS} G(z) := \sum_{k=0}^{n} \binom{n}{k} a_k b_k z^k$$

is of the form $\gamma = -\alpha\beta$ with $\alpha \in K$ and $G(\beta) = 0$. If $G(0) = 0$, then this continues to hold as long as K is not the open or closed exterior of a disk.

It is hard to disagree with Rota regarding the importance of Grace's theorem: It includes or implies numerous other results concerning the zero location of complex polynomials and has found many applications in complex analysis and other fields. For instance, it forms the basis of the geometric convolution theory which was developed by Ruscheweyh, Suffridge, and Sheil-Small (see [25,26,30–33] and, more recently, [27–29]) and it can be used to classify all linear operators which preserve the set of polynomials whose zeros lie in a given circular domain (cf. [26, Thm. 1.1], [32, Sec. 5.8], and [2]). Very recently, in an impressive series of papers [3–5], Borcea and Bränden used Grace's theorem in order to develop a unified analytic theory of multivariate polynomials with many astonishing applications.

One might argue, however, whether Rota was correct regarding the resistency of Grace's theorem at generalization. In fact, in 1976 Suffridge [33] (see also [16] for a different proof) showed the following stunning extension of Grace's theorem (without having any further information the author would guess that Rota was simply unaware of Suffridge's result).

Theorem 2 *(Suffridge). For $\lambda \in [0, \frac{2\pi}{n})$ let $\mathcal{P}_n(\lambda)$ denote the set of polynomials of degree n that have all their zeros on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ with each pair of zeros separated by an angle of at least λ. Then for*