Advances in Mathematics 288 (2016) 631–647

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

On the normality of secant varieties

Brooke Ullery

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109-1043, United States

A R T I C L E I N F O

Article history:
Received 5 August 2014
Received in revised form 9 October 2015
Accepted 13 October 2015
Available online 16 November 2015
Communicated by Ravi Vakil

Keywords:
Secant variety
Hilbert scheme
Normal variety
Positivity
Vector bundle

A B S T R A C T

In this paper, we show that the secant variety to a smooth projective variety embedded by a sufficiently positive line bundle is normal. As an application, we deduce that the secant variety to a general canonical curve of genus at least 7 is normal.

0. Introduction

The purpose of this paper is to show that the secant variety to a projective variety embedded by a sufficiently positive line bundle is a normal variety. In particular, this confirms the vision and completes the results of Vermeire in [20] and renders unconditional the results in [17,18,21], and [22].[1]

E-mail address: bullery@umich.edu.

[1] This question of the normality of the secant variety came up in 2001 when a proof was proposed by Vermeire [20]. However, in 2011, Adam Ginensky and Mohan Kumar pointed out that the proof was erroneous, as explained in Remark 4 of [18].

http://dx.doi.org/10.1016/j.aim.2015.10.025

Let

$$X \subset \mathbb{P}(H^0(X, \mathcal{L})) = \mathbb{P}^r$$

be a smooth variety over an algebraically closed field of characteristic zero, embedded by the complete linear system corresponding to a very ample line bundle $\mathcal{L}$. We define the secant variety

$$\Sigma(X, \mathcal{L}) \subset \mathbb{P}^n$$

to be the Zariski closure of the union of 2-secant lines to X in $\mathbb{P}^r$. As secant varieties are classical constructions in algebraic geometry, there has been a great deal of work done in an attempt to understand their geometry. Recently, there has been interest in determining defining equations and syzygies of secant varieties [2–4,17,18,23], motivated in part by questions in algebraic statistics [7,19] and algebraic complexity theory [12,13]. In this paper, we focus on the singularities of secant varieties, using the comprehensive geometric description developed by Bertram [1] and Vermeire [20].

If the embedding line bundle $\mathcal{L}$ is not sufficiently positive, the behavior of the singularities of $\Sigma(X, \mathcal{L})$ can be quite complicated. For example, the secant variety is generally singular along X, but if four points of X lie on a plane, then three pairs of secant lines will intersect away from X. In some cases this will create additional singularities at those intersection points on $\Sigma(X, \mathcal{L})$. In more degenerate cases, the secant variety may simply fill the whole projective space, e.g. the secant variety to any non-linear plane curve. However, if $\mathcal{L}$ is sufficiently positive, we will see that $\Sigma(X, \mathcal{L})$ will be singular precisely along X. As $\mathcal{L}$ becomes increasingly positive, it is natural to predict that the singularities of $\Sigma(X, \mathcal{L})$ will become easier to control.

We start by stating some concrete special cases of the main theorem. In the case of curves, normality of the secant variety only depends on a degree condition:

Corollary A. *Let X be a smooth projective curve of genus g and $\mathcal{L}$ a line bundle on X of degree d. If $d \geq 2g + 3$, then $\Sigma(X, \mathcal{L})$ is a normal variety.*

Moreover, in the example of canonical curves, we have a stronger result not covered by the above proposition:

Corollary B. *Let X be a smooth projective curve with Clifford index* $\mathrm{Cliff}(X) \geq 3$. *Then $\Sigma(X, \omega_X)$ is a normal variety.*

In particular, the above implies that the secant variety to a general canonical curve of genus at least 7 is normal.

More generally, we can also give a positivity condition on embeddings of higher dimensional varieties to ensure that the secant variety is normal: