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The Dirichlet problem for nonlocal operators with singular kernels: Convex and nonconvex domains

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A B S T R A C T

We study the interior regularity of solutions to the Dirichlet problem $Lu = g$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, for anisotropic operators of fractional type

$$Lu(x) = \int_0^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \frac{2u(x) - u(x+\rho\omega) - u(x-\rho\omega)}{\rho^{1+2s}}.$$

Here, a is any measure on S^{n-1} (a prototype example for L is given by the sum of one-dimensional fractional Laplacians in fixed, given directions).

When $a \in C^\infty(S^{n-1})$ and g is $C^\infty(\Omega)$, solutions are known to be C^∞ inside Ω (but not up to the boundary). However, when a is a general measure, or even when a is $L^\infty(S^{n-1})$, solutions are only known to be C^{3s} inside Ω.

We prove here that, for general measures a, solutions are $C^{1+3s-\epsilon}$ inside Ω for all $\epsilon > 0$ whenever Ω is convex. When $a \in L^\infty(S^{n-1})$, we show that the same holds in all $C^{1,1}$ domains. In particular, solutions always possess a classical first derivative.

The assumptions on the domain are sharp, since if the domain is not convex and the measure a is singular, we construct

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an explicit counterexample for which u is *not* $C^{3s+\epsilon}$ for any $\epsilon > 0$ – even if g and Ω are C^∞.

1. Introduction

Recently, a great attention in the literature has been devoted to the study of equations of elliptic type with fractional order $2s$, with $s \in (0,1)$. The leading example of the operators considered is the fractional Laplacian

$$(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy. \tag{1.1}$$

Several similarities arise between this operator and the classical Laplacian: for instance, the fractional Laplacian enjoys a "good" interior regularity theory in Hölder spaces and in Sobolev spaces (see e.g. [6]). Nevertheless, the fractional Laplacian also presents some striking difference with respect to the classical case: for example, solutions are in general not uniformly Lipschitz continuous up to the boundary (see e.g. [5,9]) and fractional harmonic functions are locally dense in C^k (see [3]), in sharp contrast with respect to the classical case.

A simple difference between the fractional and the classical Laplacians is also given by the fact that the classical Laplacian may be reconstructed as the sum of finitely many one-dimensional operators, namely one can write

$$\Delta = \partial_1^2 + \cdots + \partial_n^2, \tag{1.2}$$

and each ∂_i^2 is indeed the Laplacian in a given direction. This phenomenon is typical for the classical case and it has no counterpart in the fractional setting, since the operator in (1.1) cannot be reduced to finite sets of directions.

Nevertheless, in order to study equations in anisotropic media, it is important to understand operators obtained by the superposition of fractional one-dimensional (or lower-dimensional) operators, or, more generally, by the superposition of different operators in different directions, see [8]. For this reason, we consider here the anisotropic integro-differential operator

$$Lu(x) = \int_0^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \frac{2u(x) - u(x+\rho\omega) - u(x-\rho\omega)}{\rho^{1+2s}}, \tag{1.3}$$

with $s \in (0,1)$. Here a is a non-negative measure on S^{n-1} (called in jargon the "spectral measure"), and we suppose that it satisfies the following "ellipticity" assumption