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On the wave representation of hyperbolic, elliptic, and parabolic Eisenstein series ☆

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We develop a unified approach to the construction of the hyperbolic and elliptic Eisenstein series on a finite volume hyperbolic Riemann surface. Specifically, we derive expressions for the hyperbolic and elliptic Eisenstein series as integral transforms of the kernel of a wave operator. Established results in the literature relate the wave kernel to the heat kernel, which admits explicit construction from various points of view. Therefore, we obtain a sequence of integral transforms which begins with the heat kernel, obtains a Poisson and wave kernel, and then yields the hyperbolic and elliptic Eisenstein series. In the case of a non-compact finite volume hyperbolic Riemann surface, we finally show how to express the parabolic Eisenstein series in terms of the integral transform of a wave kernel.

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1. Introduction

1.1. Non-holomorphic Eisenstein series

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind which acts on the hyperbolic space $\mathbb{H}$ by fractional linear transformations, and let $M = \Gamma\backslash\mathbb{H}$ be the finite volume quotient. One can view M as a finite volume hyperbolic Riemann surface, possibly with p_Γ cusps and e_Γ elliptic fixed points. Classically, associated to any cusp p_j ($j = 1, \ldots, p_\Gamma$) of M, there is a non-holomorphic *parabolic Eisenstein series* defined for $z \in M$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by

$$\mathcal{E}^{\mathrm{par}}_{p_j}(z, s) = \sum_{\eta \in \Gamma_{p_j}\backslash\Gamma} \mathrm{Im}(\sigma_{p_j}^{-1}\eta z)^s, \tag{1}$$

where $\Gamma_{p_j} := \mathrm{Stab}_\Gamma(p_j) = \langle \gamma_{p_j} \rangle$ is the stabilizer subgroup generated by a primitive, parabolic element $\gamma_{p_j} \in \Gamma$ and $\sigma_{p_j} \in \mathrm{PSL}_2(\mathbb{R})$ is the scaling matrix which satisfies $\sigma_{p_j}^{-1}\gamma_{p_j}\sigma_{p_j} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The parabolic Eisenstein series is an automorphic function on M and admits a meromorphic continuation to the whole complex s-plane with no poles on the line $\mathrm{Re}(s) = 1/2$ (see, for example, [3,9,10], or [15]). The parabolic Eisenstein series plays a central role in the spectral theory of automorphic functions on M by contributing the eigenfunctions for the continuous spectrum of the hyperbolic Laplacian on M, and, subsequently, in the Selberg trace formula.

In [16], Kudla and Millson defined and studied a form-valued, non-holomorphic Eisenstein series $\mathcal{E}^{\mathrm{hyp}}_{\gamma,\mathrm{KM}}(z, s)$ associated to any simple closed geodesic of M, or equivalently to any primitive, hyperbolic element $\gamma \in \Gamma$. In analogy with results for scalar-valued, parabolic, non-holomorphic Eisenstein series, the following results were proved in [16]. First, the Eisenstein series $\mathcal{E}^{\mathrm{hyp}}_{\gamma,\mathrm{KM}}(z, s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$. Second, the special value at $s = 0$ of the meromorphic continuation is a harmonic form which is the Poincaré dual to the geodesic corresponding to the hyperbolic element $\gamma \in \Gamma$.

A scalar-valued, non-holomorphic, hyperbolic Eisenstein series was defined in [12], and the authors proved that the series admits a meromorphic continuation to all $s \in \mathbb{C}$. Let $\gamma \in \Gamma$ be a primitive, hyperbolic element with stabilizer Γ_γ. Let $\mathcal{L}_\gamma$ be the geodesic in $\mathbb{H}$ which is invariant by the action of γ on $\mathbb{H}$. Then, for $z \in M$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the *hyperbolic Eisenstein series* is defined by

$$\mathcal{E}^{\mathrm{hyp}}_{\gamma}(z, s) = \sum_{\eta \in \Gamma_\gamma\backslash\Gamma} \cosh(d_{\mathrm{hyp}}(\eta z, \mathcal{L}_\gamma))^{-s},$$

where $d_{\mathrm{hyp}}(\eta z, \mathcal{L}_\gamma)$ denotes the hyperbolic distance from the point ηz to the arc $\mathcal{L}_\gamma$. In [12], it was proved that $\mathcal{E}^{\mathrm{hyp}}_{\gamma}(z, s)$ admits an L^2-spectral expansion, which was explicitly computed, and from which the meromorphic continuation of $\mathcal{E}^{\mathrm{hyp}}_{\gamma}(z, s)$ in s was derived.