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An extended Fatou–Shishikura inequality and wandering branch continua for polynomials

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Let P be a polynomial of degree d with Julia set J_P. Let $\widetilde{N}$ be the number of non-repelling cycles of P. By the famous Fatou–Shishikura inequality $\widetilde{N} \leq d-1$. The goal of the paper is to improve this bound. The new count includes *wandering collections of non-(pre)critical branch continua*, i.e., collections of continua or points $Q_i \subset J_P$ *all* of whose images are pairwise disjoint, contain no critical points, and contain the limit sets of $\text{eval}(Q_i) \geq 3$ external rays. Also, we relate individual cycles, which are either non-repelling or repelling with no periodic rays landing, to individual critical points that are recurrent in a weak sense.
A weak version of the inequality reads

$$\widetilde{N} + N_{irr} + \chi + \sum_i (\text{eval}(Q_i) - 2) \leq d-1$$

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Julia set
Wandering continuum

where N_{irr} counts repelling cycles with no periodic rays landing at points in the cycle, $\{Q_i\}$ form a wandering collection $\mathcal{B}_{\mathbb{C}}$ of non-(pre)critical branch continua, $\chi = 1$ if $\mathcal{B}_{\mathbb{C}}$ is non-empty, and $\chi = 0$ otherwise.

1. Introduction

In the dynamics of iterated rational maps, it is a frequent observation that many interesting dynamical features are largely determined by the dynamics of critical points. The classical Fatou–Shishikura inequality states in the polynomial case that a complex polynomial of degree $d \geq 2$ has at most $d - 1$ non-repelling periodic orbits in $\mathbb{C}$. We extend this in several ways.

- *Wandering (eventual) branch continua*, defined below, are included in the count (such continua are either proper subsets of periodic components of the Julia set or wandering components of the Julia set); note that we allow continua to be points. In the simplest case, such a continuum corresponds to a point z in the Julia set that is the landing point of 3 or more external rays so that no point in the forward orbit of z is critical or periodic.
- Together with non-repelling periodic orbits, we also count orbits of repelling periodic points that are not landing points of *periodic* external rays (such points may exist if the Julia set is not connected and then must be components of the Julia set).
- Specific critical points are associated to the aforementioned periodic orbits and wandering branch continua: (a) every non-repelling periodic orbit and every repelling periodic orbit without periodic rays has at least one associated critical point, so that different orbits are associated to different critical points, and (b) wandering branch continua require other critical points not associated to any periodic orbits.
- The inequality is sharpened by counting not all critical points, but certain "weak equivalence classes of weakly recurrent critical points" (other restrictions on critical points apply as well).
- The key idea is that various phenomena counted on the left hand side of the inequality can be associated with critical points counted on the right. In the case of wandering eventual branch continua the association is not as direct as in the case of specific periodic points, but sufficient for our purpose.

Let P be a polynomial of degree $d \geq 2$ with Julia set J_P. A *rational ray pair* $\mathcal{R}$ is a pair of (pre)periodic external rays that land at a common point, together with their common landing point; $\mathcal{R}$ *weakly separates* two points $z, w \in \mathbb{C}$ if z and w are in two different components of $\mathbb{C} \setminus \mathcal{R}$. A critical point c is *weakly recurrent* if it belongs to the filled-in Julia set, never maps to a repelling or parabolic point, and for every finite