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The multiplicative eigenvalue problem and deformed quantum cohomology

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ABSTRACT

We construct deformations of the small quantum cohomology rings of homogeneous spaces G/P, and obtain an irredundant set of inequalities determining the multiplicative eigen polytope for the compact form K of G.

1. Introduction

Let G be a simple, connected, simply-connected complex algebraic group. We choose a Borel subgroup B and a maximal torus $H \subset B$. We denote their Lie algebras by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{h}$ respectively. Let $R = R_{\mathfrak{g}} \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ and let R^+ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$). Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R^+$ be the set of simple roots.

Consider the *fundamental alcove* $\mathscr{A} \subset \mathfrak{h}$ defined by

$$\mathscr{A} = \{\mu \in \mathfrak{h} : \alpha_i(\mu) \geq 0 \text{ for all simple roots } \alpha_i \text{ and } \theta_o(\mu) \leq 1\},$$

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where θ_o is the highest root of $\mathfrak{g}$. Then, $\mathscr{A}$ parameterizes the K-conjugacy classes of K under the map $C : \mathscr{A} \to K/\operatorname{Ad} K$,

$$\mu \mapsto c(\operatorname{Exp}(2\pi i\mu)),$$

where K is a maximal compact subgroup of G and $c(\operatorname{Exp}(2\pi i\mu))$ denotes the K-conjugacy class of $\operatorname{Exp}(2\pi i\mu)$. Fix a positive integer $n \geq 3$ and define the *multiplicative eigen polytope*

$$\mathscr{C}_n := \{(\mu_1, \ldots, \mu_n) \in \mathscr{A}^n : 1 \in C(\mu_1)\ldots C(\mu_n)\}.$$

Then, $\mathscr{C}_n$ is a rational convex polytope with nonempty interior in $\mathfrak{h}^n$. Our aim is to describe the facets (i.e., the codimension one faces) of $\mathscr{C}_n$ which meet the interior of $\mathscr{A}^n$.

We need to introduce some more notation before we can state our results. Let P be a standard parabolic subgroup (i.e., $P \supset B$) and let $L \subset P$ be its Levi subgroup containing H. Then, $B_L := B \cap L$ is a Borel subgroup of L. We denote the Lie algebras of P, L, B_L by the corresponding Gothic characters: $\mathfrak{p}$, $\mathfrak{l}$, $\mathfrak{b}_L$ respectively. Let $R_{\mathfrak{l}}$ be the set of roots of $\mathfrak{l}$ and $R_{\mathfrak{l}}^+$ be the set of roots of $\mathfrak{b}_L$. We denote by Δ_P the set of simple roots contained in $R_{\mathfrak{l}}$ and we set

$$S_P := \Delta \setminus \Delta_P.$$

For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}$ by

$$\alpha_i(x_j) = \delta_{i,j},\ \forall\ 1 \leq i \leq \ell.$$

Let W be the Weyl group of G and let W^P be the set of the minimal length representatives in the cosets of W/W_P, where W_P is the Weyl group of P. For any $w \in W^P$, let $X_w^P := \overline{BwP/P} \subset G/P$ be the corresponding Schubert variety and let $\{\sigma_w^P\}_{w \in W^P}$ be the Poincaré dual (dual to the fundamental class of X_w^P) basis of $H^*(G/P, \mathbb{Z})$.

We begin with the following theorem. It was proved by Biswas [14] in the case $G = \mathrm{SL}_2$; by Belkale [5] for $G = \mathrm{SL}_m$ (and in this case a slightly weaker result by Agnihotri–Woodward [1] where the inequalities were parameterized by $\langle \sigma_{u_1}^P, \ldots, \sigma_{u_n}^P \rangle_d \neq 0$); and by Teleman–Woodward [39] for general G. It may be recalled that the precursor to these results was the result due to Klyachko [23] determining the additive eigencone for SL_m.

Theorem 1.1. *Let $(\mu_1, \ldots, \mu_n) \in \mathscr{A}^n$. Then, the following are equivalent:*

(a) $(\mu_1, \ldots, \mu_n) \in \mathscr{C}_n$.

(b) For any standard maximal parabolic subgroup P of G, any $u_1, \ldots, u_n \in W^P$, and any $d \geq 0$ such that the Gromov–Witten invariant (cf. Definition 2.1)