

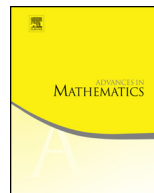


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# The multiplicative eigenvalue problem and deformed quantum cohomology



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## ABSTRACT

We construct deformations of the small quantum cohomology rings of homogeneous spaces  $G/P$ , and obtain an irredundant set of inequalities determining the multiplicative eigen polytope for the compact form  $K$  of  $G$ .

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## 1. Introduction

Let  $G$  be a simple, connected, simply-connected complex algebraic group. We choose a Borel subgroup  $B$  and a maximal torus  $H \subset B$ . We denote their Lie algebras by  $\mathfrak{g}$ ,  $\mathfrak{b}$ ,  $\mathfrak{h}$  respectively. Let  $R = R_{\mathfrak{g}} \subset \mathfrak{h}^*$  be the set of roots of  $\mathfrak{g}$  and let  $R^+$  be the set of positive roots (i.e., the set of roots of  $\mathfrak{b}$ ). Let  $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R^+$  be the set of simple roots.

Consider the *fundamental alcove*  $\mathcal{A} \subset \mathfrak{h}$  defined by

$$\mathcal{A} = \{\mu \in \mathfrak{h} : \alpha_i(\mu) \geq 0 \text{ for all simple roots } \alpha_i \text{ and } \theta_o(\mu) \leq 1\},$$

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where  $\theta_o$  is the highest root of  $\mathfrak{g}$ . Then,  $\mathcal{A}$  parameterizes the  $K$ -conjugacy classes of  $K$  under the map  $C : \mathcal{A} \rightarrow K/\text{Ad } K$ ,

$$\mu \mapsto c(\text{Exp}(2\pi i\mu)),$$

where  $K$  is a maximal compact subgroup of  $G$  and  $c(\text{Exp}(2\pi i\mu))$  denotes the  $K$ -conjugacy class of  $\text{Exp}(2\pi i\mu)$ . Fix a positive integer  $n \geq 3$  and define the *multiplicative eigen polytope*

$$\mathcal{C}_n := \{(\mu_1, \dots, \mu_n) \in \mathcal{A}^n : 1 \in C(\mu_1) \dots C(\mu_n)\}.$$

Then,  $\mathcal{C}_n$  is a rational convex polytope with nonempty interior in  $\mathfrak{h}^n$ . Our aim is to describe the facets (i.e., the codimension one faces) of  $\mathcal{C}_n$  which meet the interior of  $\mathcal{A}^n$ .

We need to introduce some more notation before we can state our results. Let  $P$  be a standard parabolic subgroup (i.e.,  $P \supset B$ ) and let  $L \subset P$  be its Levi subgroup containing  $H$ . Then,  $B_L := B \cap L$  is a Borel subgroup of  $L$ . We denote the Lie algebras of  $P, L, B_L$  by the corresponding Gothic characters:  $\mathfrak{p}, \mathfrak{l}, \mathfrak{b}_L$  respectively. Let  $R_{\mathfrak{l}}$  be the set of roots of  $\mathfrak{l}$  and  $R_{\mathfrak{l}}^+$  be the set of roots of  $\mathfrak{b}_L$ . We denote by  $\Delta_P$  the set of simple roots contained in  $R_{\mathfrak{l}}$  and we set

$$S_P := \Delta \setminus \Delta_P.$$

For any  $1 \leq j \leq \ell$ , define the element  $x_j \in \mathfrak{h}$  by

$$\alpha_i(x_j) = \delta_{i,j}, \quad \forall 1 \leq i \leq \ell.$$

Let  $W$  be the Weyl group of  $G$  and let  $W^P$  be the set of the minimal length representatives in the cosets of  $W/W_P$ , where  $W_P$  is the Weyl group of  $P$ . For any  $w \in W^P$ , let  $X_w^P := \overline{BwP/P} \subset G/P$  be the corresponding Schubert variety and let  $\{\sigma_w^P\}_{w \in W^P}$  be the Poincaré dual (dual to the fundamental class of  $X_w^P$ ) basis of  $H^*(G/P, \mathbb{Z})$ .

We begin with the following theorem. It was proved by Biswas [14] in the case  $G = \text{SL}_2$ ; by Belkale [5] for  $G = \text{SL}_m$  (and in this case a slightly weaker result by Agnihotri–Woodward [1] where the inequalities were parameterized by  $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d \neq 0$ ); and by Teleman–Woodward [39] for general  $G$ . It may be recalled that the precursor to these results was the result due to Klyachko [23] determining the additive eigencone for  $\text{SL}_m$ .

**Theorem 1.1.** *Let  $(\mu_1, \dots, \mu_n) \in \mathcal{A}^n$ . Then, the following are equivalent:*

- (a)  $(\mu_1, \dots, \mu_n) \in \mathcal{C}_n$ .
- (b) For any standard maximal parabolic subgroup  $P$  of  $G$ , any  $u_1, \dots, u_n \in W^P$ , and any  $d \geq 0$  such that the Gromov–Witten invariant (cf. Definition 2.1)

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