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Dynamical anomalous subvarieties: Structure and bounded height theorems ☆

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ABSTRACT

According to Medvedev and Scanlon [14], a polynomial $f(x) \in \bar{\mathbb{Q}}[x]$ of degree $d \geq 2$ is called disintegrated if it is not conjugate to x^d or to $\pm C_d(x)$ (where C_d is the Chebyshev polynomial of degree d). Let $n \in \mathbb{N}$, let $f_1, \ldots, f_n \in \bar{\mathbb{Q}}[x]$ be disintegrated polynomials of degrees at least 2, and let $\varphi = f_1 \times \cdots \times f_n$ be the corresponding coordinate-wise self-map of $(\mathbb{P}^1)^n$. Let X be an irreducible subvariety of $(\mathbb{P}^1)^n$ of dimension r defined over $\bar{\mathbb{Q}}$. We define the φ-*anomalous* locus of X which is related to the φ-*periodic* subvarieties of $(\mathbb{P}^1)^n$. We prove that the φ-anomalous locus of X is Zariski closed; this is a dynamical analogue of a theorem of Bombieri, Masser, and Zannier [4]. We also prove that the points in the intersection of X with the union of all irreducible φ-periodic subvarieties of $(\mathbb{P}^1)^n$ of codimension r have bounded height outside the φ-anomalous locus of X; this is a dynamical analogue of Habegger's theorem [8] which was previously conjectured in [4]. The slightly more general self-maps $\varphi = f_1 \times \cdots \times f_n$ where each $f_i \in \bar{\mathbb{Q}}(x)$ is a disintegrated rational function are also treated at the end of the paper.

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1. Introduction

Throughout this paper, a variety is always over $\bar{\mathbb{Q}}$ and is defined as in [1, Appendix A.4]. In other words, a variety is the set of closed points of a (not necessarily irreducible) reduced and separated scheme of finite type over $\bar{\mathbb{Q}}$ equipped with the Zariski topology, sheaf of regular functions, etc. For a map μ from a set to itself and for every positive integer m, we let μ^m denote the m-fold iterate: $\mu \circ \ldots \circ \mu$; μ^0 denotes the identity map. Let h denote the absolute logarithmic Weil height on $\mathbb{P}^1$ (see [1, Chapter 1] or [10, Part B]). Let n be a positive integer, we define the height function h_n on $(\mathbb{P}^1)^n$ by $h_n(a_1, \ldots, a_n) := h(a_1) + \ldots + h(a_n)$. When we say that a subset of $(\mathbb{P}^1)^n$ has bounded height, we mean boundedness with respect to h_n.

After a series of papers [3,5,4] following the seminal work [2, Theorem 1], Bombieri, Masser and Zannier define anomalous subvarieties in $\mathbb{G}_{\mathrm{m}}^n$ as follows. By a *special* subvariety of $\mathbb{G}_{\mathrm{m}}^n$, we mean a translate of an irreducible algebraic subgroup. For any irreducible subvariety $X \subseteq \mathbb{G}_{\mathrm{m}}^n$ of dimension r, an irreducible subvariety Y of X is said to be *anomalous* (or better, X-anomalous) if there exists a special subvariety Z satisfying the following conditions:

$$Y \subseteq X \cap Z \text{ and } \dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}. \tag{1}$$

We define $X^{\mathrm{oa}} := X \setminus \bigcup_Y Y$, where Y ranges over all anomalous subvarieties of X. We let $\mathbb{G}_m^{n,[r]}$ be the union of all algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n$ of *codimension* r. The following has been established by Bombieri, Masser, Zannier [4, Theorem 1.4] and Habegger [8, Theorem 1.2] (after being previously conjectured in [4]):

Theorem 1.1. *Let X be an irreducible subvariety of $\mathbb{G}_m^n$ of dimension r (defined over $\bar{\mathbb{Q}}$, as always). We have:*

(a) (Bombieri–Masser–Zannier) *Structure Theorem: the set X^{oa} is Zariski open in X. Moreover, there exists a finite collection $\mathcal{T}$ of subtori of $\mathbb{G}_m^n$ (depending on X) such that the anomalous locus of X is the union of all anomalous subvarieties Y of X for which there exists a translate Z of a tori in $\mathcal{T}$ satisfying $Y \subseteq X \cap Z$ and* $\dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}$.

(b) (Habegger) *Bounded Height Theorem: the set $X^{\mathrm{oa}} \cap \mathbb{G}_m^{n,[r]}$ has bounded height.*

The Bounded Height Theorem is closely related to the problem of unlikely intersections in arithmetic geometry introduced in [2] whose motivation comes from the classical Manin–Mumford conjecture (which is Raynaud's theorem [18,19] for abelian varieties