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Asymptotic completeness for the massless spin-boson model

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We consider generalized versions of the massless spin-boson model. Building on the recent work in [3] and [4], we prove asymptotic completeness.

1. Introduction

This paper is concerned with the scattering theory for generalized spin-boson models with massless bosons. That is, we consider a spin system (an "atom") coupled to a scalar field of quantized massless bosons. With the help of previously established properties, such as relaxation to the ground state and a uniform bound on the number of soft bosons, we now show that asymptotic completeness holds provided the excited states of

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the uncoupled system of spin and bosons have finite life times once the interaction is turned on. (Fermi-Golden Rule condition.)

To describe our main result and its proof we now introduce the system in some detail. We confine ourselves to a concrete system satisfying all our assumptions. More general hypotheses are described in the next section. Our model consists of a small system (atom, spin) coupled to a free bosonic field. The Hilbert space of the total system is

$$\mathscr{H} = \mathscr{H}_{\mathrm{S}} \otimes \mathscr{H}_{\mathrm{F}}$$

where $\mathscr{H}_{\mathrm{S}} = \mathbb{C}^n$ for some $n < \infty$ (S for "small system") and the field space $\mathscr{H}_{\mathrm{F}}$ is the symmetric Fock space over $L^2(\mathbb{R}^3)$. The total Hamiltonian is of the form

$$H = H_{\mathrm{S}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{F}} + H_{\mathrm{I}}$$

where H_{S} is a hermitian matrix with simple eigenvalues only, and H_{F} denotes the Hamiltonian of the free *massless* field. The coupling operator H_{I} is of the form

$$H_{\mathrm{I}} = \lambda D \otimes \int_{\mathbb{R}^3} \big(\hat{\phi}(k) a_k^* + \overline{\hat{\phi}(k)} a_k\big) \mathrm{d}k.$$

Here $D = D^*$ is a matrix acting on $\mathscr{H}_{\mathrm{S}}$, $\lambda \in \mathbb{R}$ is a sufficiently small coupling constant, and a_k^*, a_k are the usual creation and annihilation operators of a mode $k \in \mathbb{R}^3$ satisfying the "Canonical Commutation Relations". The function $\hat{\phi} \in L^2(\mathbb{R}^3)$ is a "form factor" that imposes some infrared regularity and an ultraviolet cutoff: simple and sufficient assumptions are that $\hat{\phi}$ has compact support and that $\hat{\phi} \in C^3(\mathbb{R}^3 \backslash \{0\})$ with

$$\hat{\phi}(k) = |k|^{(\alpha-1)/2}, \quad |k| \le 1,$$

for some $\alpha > 0$. This assumption ensures, e.g., that H has a unique ground state Ψ_{gs}. To rule out the existence of excited bound states we assume the Fermi-Golden Rule condition stated in the next section. Further spectral input is not needed but our results do certainly have non-trivial consequences for the spectrum of H.

Under the time-evolution generated by H it is expected that every excited state relaxes to the ground state by emission of photons whose dynamics is asymptotically free. The existence of excited states with this property is well known [9,16]; they are spanned by products of asymptotic creation operators applied to the ground state Ψ_{gs}, that is, by vectors of the form:

$$a_+^*(f_1) \dots a_+^*(f_m) \Psi_{\mathrm{gs}} = \lim_{t\to\infty} e^{i(H-E)t} a^*(f_{1,t}) \dots a^*(f_{m,t}) \Psi_{\mathrm{gs}},$$

where $f_t = e^{-i\omega t} f$ and $\omega(k) = |k|$. Asymptotic completeness of Rayleigh scattering means that the span of these vectors is dense in $\mathscr{H}$. In particular, the representation