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Euler flag enumeration of Whitney stratified spaces

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ABSTRACT

The flag vector contains all the face incidence data of a polytope, and in the poset setting, the chain enumerative data. It is a classical result due to Bayer and Klapper that for face lattices of polytopes, and more generally, Eulerian graded posets, the flag vector can be written as a **cd**-index, a non-commutative polynomial which removes all the linear redundancies among the flag vector entries. This result holds for regular CW complexes.

We relax the regularity condition to show the **cd**-index exists for Whitney stratified manifolds by extending the notion of a graded poset to that of a quasi-graded poset. This is a poset endowed with an order-preserving rank function and a weighted zeta function. This allows us to generalize the classical notion of Eulerianness, and obtain a **cd**-index in the quasi-graded poset arena. We also extend the semi-suspension operation to that of embedding a complex in the boundary of a higher dimensional ball and study the simplicial shelling components.

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1. Introduction

In this paper we extend the theory of face incidence enumeration of polytopes, and more generally, chain enumeration in graded Eulerian posets, to that of Whitney stratified spaces and quasi-graded posets.

The idea of enumeration using the Euler characteristic was suggested throughout Rota's work and influenced by Schanuel's categorical viewpoint [33,40–42]. In order to carry out such a program that is topologically meaningful and which captures the broadest possible classes of examples, two key insights are required. First, the notion of grading in the face lattice of a polytope must be relaxed. Secondly, the usual zeta function in the incidence algebra must be extended to include the Euler characteristic as an important instance.

Recall an *Eulerian poset* is a graded partially ordered set (poset) P such that every nontrivial interval satisfies the Euler–Poincaré relation, that is, the number of elements of even rank equals the number of elements of odd rank. Equivalently, the Möbius function is given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$, where ρ denotes the *rank function*. Another way to state this *Eulerian condition* is that the inverse of the zeta function $\zeta(x, y)$ (where $\zeta(x, y) = 1$ if $x \leq y$ and 0 otherwise), that is, the Möbius function $\mu(x, y)$, is given by the function $(-1)^{\rho(y)-\rho(x)} \cdot \zeta(x, y)$. Families of Eulerian posets include (a) the face lattice of a convex polytope, (b) the face poset of a regular cell decomposition of a homology sphere, and (c) the elements of a finite Coxeter group ordered by the strong Bruhat order. In the case of a convex polytope the Eulerian condition expresses the fact that the link of each face has the Euler characteristic of a sphere.

The *f-vector* of a convex polytope enumerates, for each non-negative integer i, the number f_i of i-dimensional faces in the polytope. It satisfies the Euler–Poincaré relation. The problem of understanding the f-vectors of polytopes harks back to Steinitz [49], who completely described the 3-dimensional case. For polytopes of dimension greater than three the problem is still open. For *simplicial* polytopes, that is, each i-dimensional face is an i-dimensional simplex, the f-vectors satisfy linear relations known as the Dehn–Sommerville relations. Furthermore, the f-vectors of simplicial polytopes have been completely characterized by work of McMullen [37], Billera and Lee [7], and Stanley [44].

The *flag f-vector* of a graded poset counts the number of chains passing through a prescribed set of ranks. In the case of a polytope, it records all of the face incidence data, including that of the f-vector. Bayer and Billera proved that the flag f-vector of any Eulerian poset satisfies a collection of linear equalities now known as the *generalized Dehn–Sommerville relations* [2]. These linear equations may be interpreted as natural redundancies among the components of the flag f-vector. Bayer and Klapper removed these redundancies by showing that the space of flag f-vectors of Eulerian posets has a natural basis consisting of non-commutative polynomials in the two variables **c** and **d** [3]. The coefficients of this **cd**-*index* were later shown by Stanley to be non-negative in the case of spherically-shellable posets [46]. Other milestones for the **cd**-index include