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On quantizations of complex contact manifolds

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ABSTRACT

A (holomorphic) quantization of a complex contact manifold is a filtered algebroid stack which is locally equivalent to the ring $\mathcal{E}$ of microdifferential operators and which has trivial graded. The existence of a canonical quantization has been proved by Kashiwara. In this paper we consider the classification problem, showing that the above quantizations are classified by the first cohomology group with values in a certain sheaf of homogeneous forms. Secondly, we consider the problem of existence and classification for quantizations given by algebras.

0. Introduction

This paper deals with the problem of existence and classification for the quantizations of complex contact manifolds.

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Precisely, consider a complex contact manifold (Y, α), *i.e.* a complex manifold Y endowed with a projective 1-form α (a global section of the projective cotangent bundle $P^*Y \to Y$) such that $d\alpha|_H$ is non-degenerate, where H denotes the codimension 1 subbundle of the tangent bundle associated to α via projective duality. Let $\mathcal{L} = \alpha^*\mathcal{O}_{P^*Y}(1)$ be the pull-back of the relative Serre sheaf on P^*Y: it is a locally free $\mathcal{O}_Y$-module of rank 1 endowed with a Lie bracket induced by α (the Lagrange bracket). Finally, denote by $\mathcal{S}_Y$ the sheaf on Y whose local sections are symbols, *i.e.* series $\sum_{j=-\infty}^{m} f_j$ with $f_j \in \mathcal{L}^{\otimes j}$ and satisfying some growth conditions. $\mathcal{S}_Y$ is endowed with a natural filtration and the associated graded sheaf is the "homogeneous coordinates ring" $\mathcal{O}_Y^{hom} = \bigoplus_{m\in\mathbb{Z}} \mathcal{L}^{\otimes m}$.

A (holomorphic) quantization algebra on (Y, α) is a sheaf of algebras $\mathcal{A}$ on Y locally isomorphic to $\mathcal{S}_Y$ as $\mathbb{C}_Y$-modules, in such a way that the product on $\mathcal{S}_Y$ induced by that of $\mathcal{A}$ is a formal bidifferential operator and is compatible both with the algebra structure on $\mathcal{O}_Y$ and with the Lie bracket on $\mathcal{L}$. Such $\mathcal{A}$ is naturally filtered with $\mathcal{O}_Y^{hom}$ as graded sheaf and the above compatibility conditions mean that $\mathrm{Gr}(\mathcal{A}) \simeq \mathcal{O}_Y^{hom}$ is a graded algebra isomorphism commuting with the Poisson structures (induced by the commutator on the left-hand side and by the Lagrange bracket on the right-hand side).

Darboux's theorem for complex contact manifolds asserts that for any point $y \in Y$ there exists an open neighborhood V of y and a contact transformation $i: V \to P^*M$ for some complex manifold M (here P^*M is endowed with the canonical contact structure given by the Liouville 1-form). Since any quantization algebra on V is locally isomorphic through i to the $\mathbb{C}$-algebra $\mathcal{E}_{P^*M}$ of Sato's microdifferential operators (a localization of the algebra $\mathcal{D}_M$ of differential operators on M), it follows that the quantization algebras on Y are nothing but $\mathcal{E}$-algebras, *i.e.* $\mathbb{C}$-algebras locally isomorphic to $i^{-1}\mathcal{E}_{P^*M}$ for any Darboux chart $i: Y \supset V \to P^*M$.

In the case $Y = P^*M$, formal (*i.e.* without growth conditions) $\mathcal{E}$-algebras were classified by Boutet de Monvel in [3].

For a complex contact manifold Y, the situation is more complicated, since $\mathcal{E}$-algebras may not exist globally. However Kashiwara in [19] proved that there exists a canonical stack (sheaf of categories) of modules over locally defined $\mathcal{E}$-algebras. In fact, this stack is equivalent to the stack of modules over an algebroid stack E_Y (see [23,9]). Moreover, E_Y has the same properties of an $\mathcal{E}$-algebra: it is filtered, locally equivalent to an $\mathcal{E}$-algebra and it has $\mathcal{O}_Y^{hom}$ as associated graded stack.

Hence, it makes sense to say that E_Y is a (holomorphic) quantization of Y and to replace $\mathcal{E}$-algebras by $\mathbb{C}$-linear stacks locally equivalent to E_Y. Being locally modeled on $\mathcal{E}$-algebras, these objects have locally trivial graded, but their associated graded stack in general is non-trivial. Thus, we are lead to define a $(\mathcal{E}, \sigma)$-algebroid, playing the role of quantization of Y, as a filtered $\mathbb{C}$-linear stacks which is locally equivalent to E_Y and which has $\mathcal{O}_Y^{hom}$ as associated graded stack.

In this paper we classify all $(\mathcal{E}, \sigma)$-algebroids on any complex contact manifold Y. This is done by means of the cohomology group $H^1(Y; \Omega_Y^{1;cl}(0))$, where $\Omega_Y^{1;cl}(0)$ denotes the push-forward of the sheaf of closed 0-homogeneous 1-forms on the canonical