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Discrete approximations for complex Kac–Moody groups

John D. Foley [1,2]

Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, USA

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ABSTRACT

We construct a map from the classifying space of a discrete Kac–Moody group over the algebraic closure of the field with p elements to the classifying space of a complex topological Kac–Moody group and prove that it is a homology equivalence at primes q different from p. This generalizes a classical result of Quillen, Friedlander and Mislin for Lie groups. As an application, we construct unstable Adams operations for general Kac–Moody groups compatible with the Frobenius homomorphism. Our results rely on new integral homology decompositions for certain infinite dimensional unipotent subgroups of discrete Kac–Moody groups.

E-mail address: foley@math.ku.dk.
[1] Permanent address: Department of Mathematics, University of Copenhagen, 2100 København Ø, Denmark.
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1. Introduction

Cohomological approximations for Lie groups by related discrete groups were developed by Quillen [44], Milnor [40], Friedlander and Mislin [23]. In this paper, we prove that a complex Kac–Moody group is cohomologically approximated by the corresponding discrete Kac–Moody group over $\bar{\mathbb{F}}_p$ at primes q different from p. One application is the construction of unstable Adams operations for arbitrary complex Kac–Moody groups.

Over the complex numbers, topological Kac–Moody groups are constructed by integrating Kac–Moody Lie algebras [31] that are typically infinite dimensional, but integrate to Lie groups when finite dimensional. Kac–Moody Lie algebras are defined via generators and relations encoded in a generalized Cartan matrix [38]. Kac–Moody groups K have a finite rank maximal torus that is unique up to conjugation by a Coxeter Weyl group; this Weyl group is finite exactly when K is Lie. For any minimal split topological K [32,38], Tits [49] constructed a corresponding discrete Kac–Moody group functor $K(-)$ from the category of commutative rings with unit to the category of groups such that $K = K(\mathbb{C})$ as abstract groups. In this paper, a Kac–Moody group is a value of such a functor applied to some fixed ring or the corresponding connected topological group [32,38].

Theorem A. *Let K be a topological complex Kac–Moody group and let $K(\bar{\mathbb{F}}_p)$ be the value of the corresponding discrete Kac–Moody group functor. Then, there exists a map $BK(\bar{\mathbb{F}}_p) \to BK$ that is an isomorphism on homology with $\mathbb{F}_q$ coefficients for any $q \neq p$.*

Theorem A is proved by extending Friedlander and Mislin's map [23, Theorem 1.4] between the classifying spaces of appropriate reductive subgroups of discrete and topological Kac–Moody groups to a map between the full classifying spaces of Kac–Moody groups. This extension uses a new homology decomposition of a discrete Kac–Moody group over a field away from the ambient characteristic. We state this result now, but see Theorem 4.1.1 for a more precise statement.

Theorem B. *Let $K(\mathbb{F})$ be a Kac–Moody group over a field. Then there is a finite collection of subgroups $\{G_I(\mathbb{F})\}_{I\in\mathbf{S}}$ that are the $\mathbb{F}$-points of reductive algebraic groups such that the inclusions $G_I(\mathbb{F}) \hookrightarrow K(\mathbb{F})$ induce a homology equivalence*

$$\operatorname{hocolim}_{I\in\mathbf{S}} BG_I(\mathbb{F}) \longrightarrow BK(\mathbb{F}), \tag{1}$$

away from the characteristic of $\mathbb{F}$.

This gives a natural way to propagate cohomological approximations of complex reductive Lie groups to Kac–Moody groups; see also Remark 4.1.2.

Theorem B in turn depends on a homological vanishing result for key infinite dimensional unipotent subgroups of discrete Kac–Moody groups over fields. As explained in