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Geometry of pure spinor formalism in superstring theory

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Projective embedding of an isotropic Grassmannian $\mathrm{OGr}^+(5,10)$ into projective space of spinor representation S can be characterized with a help of Γ-matrices by equations $\Gamma^i_{\alpha\beta}\lambda^\alpha\lambda^\beta = 0$. A polynomial function of degree N with values in S defines a map to $\mathrm{OGr}^+(5,10)$ if its coefficients satisfy a $2N+1$ quadratic equations. Algebra generated by coefficients of such polynomials is a coordinate ring of the quantum isotropic Grassmannian. We show that this ring is based on a lattice; its defining relations satisfy straightened law. This enables us to compute the Poincaré series of the ring.

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1. Introduction

Complex Isotropic Grassmannian $\mathrm{OGr}^+(5, 10)$ or a space of pure spinors, as it is known in the physics literature, is a cornerstone of manifestly Poincaré covariant formulation of string theory [5]. A rigorous construction of the Hilbert space of string theory in the formalism of pure spinors remains a challenging problem (cf. [2]). Paper [1] investigates a simplified model with a quadric as the target. In [28] we proved rigorously most of the results of [1].

The paper [1] concerns certain algebra-geometric properties of the space of smooth maps $\mathrm{Map}(S^1, Q) \subset \mathrm{Map}(S^1, V^{2n})$ from a circle to a nondegenerate affine complex quadric Q in $2n$-dimensional complex linear space V^{2n}. In [28] we observed that analysis of [1] becomes rigorous if we replace $\mathrm{Map}(S^1, Q)$ by the space of polynomial maps

$$\sum_{N \leqslant k \leqslant N'} \sum_{\boldsymbol{s} \in \mathrm{G}_n} g^k_{\boldsymbol{s}} v_{\boldsymbol{s}} z^k, \quad z \in S^1 \subset \mathbb{C}^\times$$

written in some basis $(v_{\boldsymbol{s}})$, $\boldsymbol{s} \in \mathrm{G}_n = \{1, \dots, n, 1^*, \dots, n^*\}$ in V^{2n} and then pass to the limit $N' \to \infty$, $N \to -\infty$. In this basis the $\mathrm{SO}(2n)$-invariant quadratic form q splits: $q = \sum_{i=1}^n x_i x_{i^*}$. An important technical observation, on which hinge all other results in [28], is that the algebra generated by $g^k_{\boldsymbol{s}}$ is an algebra with straightened law and, as a byproduct, it is Koszul. Our goal is to prove straightened law and koszulity when Q gets replaced by $\mathrm{OGr}^+(5, 10)$. We denote corresponding coordinate rings by $A^{N'}_N(Q)$ and $A^{N'}_N(\mathrm{OGr}^+(5, 10))$. We shall refer to $\mathit{Proj}(A^{N'}_N(Q))$ and $\mathit{Proj}(A^{N'}_N(\mathrm{OGr}^+(5, 10)))$ as Quantum Quadric and Quantum Isotropic Grassmannian.

Before we plunge into rigorous analysis of the algebra $A^{N'}_N(\mathrm{OGr}^+(5, 10))$ let us briefly outline some physical applications of its Koszul property established in this paper. Recall that we started this section with an example of a quadric Q. Success with characterization of the Hilbert space $\mathcal{H}(Q)$ [1,28] depended heavily on the existence of a free graded commutative algebra resolution of $A^{N'}_N(Q)$. Is a trivial matter to construct such a resolution because $\mathit{Spec}(A^{N'}_N(Q))$ is a complete intersection and the standard Koszul complex does the job. Things become more complicated when Q gets replaced by $\mathrm{OGr}^+(5, 10)$ because the latter is not a complete intersection. Largely because of that it is still a challenge to give a rigorous description of $\mathcal{H}(\mathrm{OGr}^+(5, 10))$, which is one of the fundamental object in covariant string theory. The reader can consult [2] for possible approaches. One of them relies, as is the case of a quadric, on a resolution of $A^{N'}_N(\mathrm{OGr}^+(5, 10))$. The method for constructing such a resolution, which is used widely in physics, goes back to Koszul [20] and Tate [35]. It relies on analysis of constraint reducibilities. It was implemented in [12]