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On complete intersections with trivial canonical class

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ABSTRACT

We prove birational boundedness results on complete intersections with trivial canonical class of base point free divisors in (some version of) Fano varieties. Our results imply in particular that Batyrev–Borisov toric construction produces only a bounded set of Hodge numbers in any given dimension, even as the codimension is allowed to grow.

1. Introduction

Calabi–Yau manifolds are fascinating algebraic varieties which have been actively studied for the last 30 years. Interest in Calabi–Yau manifolds comes from at least two sources. First, these are higher dimensional analogs of the widely studied elliptic curves and K3 surfaces. Second, they have been prominently featured in string theory as possible target spaces. In string theory applications, the case of dimension three is especially relevant and much work has been done on constructing Calabi–Yau threefolds, see for example [2,5,8,9,13,16,18,20,22–24].

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There is currently no classification of Calabi–Yau threefolds, nor does it appear to be within reach. In fact, it is not known whether they form a bounded set or even if their Hodge numbers are bounded, apart from the elliptically fibered case [12]. This paper shows that one of the most prolific constructions of Calabi–Yau threefolds can only produce a birationally bounded set of them.

Known constructions of Calabi–Yau varieties fall into two general categories: "toric" and "sporadic". We are mostly concerned with the former. Batyrev's construction [2] of Calabi–Yau threefolds as crepant resolutions of generic hypersurfaces in Gorenstein Fano toric fourfolds yields the vast majority of known examples. Kreuzer and Skarke's classification [20] of Gorenstein toric Fano fourfolds yielded 473,800,776 potentially different families of Calabi–Yau threefolds, of which at least 30,108 are provably different, as measured by the Hodge number invariants.

A generalization of Batyrev's hypersurface construction, known as Batyrev–Borisov construction [6], considers generic complete intersections of k hypersurfaces in Gorenstein toric Fano varieties of dimension $k + 3$. While for any k one is guaranteed to have only a finite number of families, it has been long expected that for k large enough these complete intersections will not produce essentially new varieties. However, this has never been proved, and our paper establishes this (and in fact a stronger) result which we will state below after making some preliminary definitions.

We call a variety a bpf-big Fano (see Definition 2.4) if it is normal with Gorenstein canonical singularities and base point free and big anticanonical divisor. Also recall that a set of varieties $\mathfrak{S}$ is called birationally bounded if there exists a projective morphism $Z \to T$ with T of finite type so that for every $X \in \mathfrak{S}$ there exists $t \in T$ with fiber Z_t birational to X.

We can now state the main result of this paper.

Theorem 5.2. *Consider the set of n-dimensional varieties X_0 with Gorenstein canonical singularities and trivial canonical class which can be written as a connected component of a scheme-theoretic intersection of base point free divisors $F_i \in |D_i|$ in an $(n + k)$-dimensional bpf-big Fano variety P with $\sum_{i=1}^{k} D_i = -K_P$. Then this set is birationally bounded for any n irrespective of k.*

The proof of the above theorem is based on two main ideas. First, we use difficult results of [14] to prove that the space of bpf-big Fano varieties with Gorenstein canonical singularities with base point free anticanonical class is birationally bounded in any dimension. Then we show that a connected component of a generic (weak) Calabi–Yau complete intersection of dimension n in an $n + k$ dimensional bpf-big Fano variety can be realized as a connected component of a complete intersection in $n + l$ dimensional variety with $l \leq n$. This allows us to prove birational boundedness of (components of) generic complete intersections of this type. Finally, we pass from the generic case to the special one by finding a big base point free divisor of bounded volume.