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Rational functions with identical measure of maximal entropy

Hexi Ye

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ABSTRACT

We discuss when two rational functions f and g can have the same measure of maximal entropy. The polynomial case was completed by Beardon, Levin, Baker–Eremenko, Schmidt–Steinmetz, etc., 1980s–1990s, and we address the rational case following Levin and Przytycki (1997). We show: $\mu_f = \mu_g$ implies that f and g share an iterate ($f^n = g^m$ for some n and m) for general f with degree $d \geq 3$. And for generic $f \in \mathrm{Rat}_{d\geq 3}$, $\mu_f = \mu_g$ implies $g = f^n$ for some $n \geq 1$. For generic $f \in \mathrm{Rat}_2$, $\mu_f = \mu_g$ implies that $g = f^n$ or $\sigma_f \circ f^n$ for some $n \geq 1$, where $\sigma_f \in PSL_2(\mathbb{C})$ permutes two points in each fiber of f. Finally, we construct examples of f and g with $\mu_f = \mu_g$ such that $f^n \neq \sigma \circ g^m$ for any $\sigma \in PSL_2(\mathbb{C})$ and $m, n \geq 1$.

1. Introduction

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function with degree $d_f \geq 2$, where the projective space $\mathbb{P}^1$ is defined over $\mathbb{C}$. There is a unique probability measure μ_f on $\mathbb{P}^1$, which is invariant under f and has support equal to the Julia set J_f of f, achieving maximal entropy $h_{\mu_f} = \log d$ among all the f-invariant probability measures; see [16], [11] and [18].

In this article, we study rational functions with the same measure of maximal entropy. It is well known that $\mu_f = \mu_{f^n}$ for all iterates f^n of f, and commuting rational functions have common measure of maximal entropy. In the polynomial case, having the same measure of maximal entropy is equivalent to having the same Julia set. During the 1980s

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and 1990s, pairs of polynomials with identical Julia set were characterized; see [22,4,5,3]. The strongest result is: given any Julia set J of some non-exceptional polynomial, there is a polynomial p, such that the set of all polynomials with Julia set J is

$$\{\sigma \circ p^n \mid n \geq 1 \text{ and } \sigma \in \Sigma_J\}, \tag{1.1}$$

where Σ_J is the set of complex affine maps on $\mathbb{C}$ preserving J. By definition, a rational function is **exceptional** if it is conformally conjugate to either a power map, $\pm$ Chebyshev polynomial, or a Lattès map. From (1.1), if f and g are two non-exceptional polynomials with $\mu_f = \mu_g$, then there exists $\sigma(z) = az + b$ preserving μ_f with

$$f^n = \sigma \circ g^m \quad \text{for some } m, n \geq 2. \tag{1.2}$$

However, unlike in the polynomial case, there exist non-exceptional rational functions with the same maximal measure but not related by the formula (1.2).

Theorem 1.1. *There exist non-exceptional rational functions f and g with degrees ≥ 2 and $\mu_f = \mu_g$, but*

$$f^n \neq \sigma \circ g^m \quad \textit{for any } \sigma \in PSL_2(\mathbb{C}) \textit{ and } n, m \geq 1. \tag{1.3}$$

Specifically, for R, S, T being rational functions with degrees ≥ 2 such that

- *For any $\sigma \in PSL_2(\mathbb{C})$, we have $R \neq \sigma \circ S$.*
- *$T \circ R = T \circ S$.*

we set $f = R \circ T$ and $g = S \circ T$, then $\mu_f = \mu_g$ and they satisfy (1.3).

The existence of the triples (R, S, T) in Theorem 1.1 is equivalent to the existence of an irreducible component of

$$V_T = \{(x, y) : T(x) = T(y)\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

with bidegree (r, r), $r \geq 2$, and normalization of genus 0. Explicit examples for such triples (R, S, T) are provided in Section 3.

Let Rat_d be the set of all rational functions with degree $d \geq 2$. The space Rat_d sits inside $\mathbb{P}^{2d-1}(\mathbb{C})$, and it is the complement of the zero locus of an irreducible homogeneous polynomial (the resultant) on $\mathbb{P}^{2d-1}$; therefore Rat_d is an affine variety. For any rational function $f \in \mathrm{Rat}_d$, denote by M_f the set of all rational functions with the same maximal entropy measure as f. As we discussed before, when f is non-exceptional and conjugate to some polynomial, M_f has very simple expression as in (1.1) by Corollary 3.2. However, from Theorem 1.1, we do not have the conclusion of (1.2) for all non-exceptional rational