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Scaling scenery of $(\times m, \times n)$ invariant measures ☆

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ABSTRACT

We study the scaling scenery and limit geometry of invariant measures for the non-conformal toral endomorphism $(x, y) \mapsto (mx \bmod 1, ny \bmod 1)$ that are Bernoulli measures for the natural Markov partition. We show that the statistics of the scaling can be described by an ergodic CP-chain in the sense of Furstenberg. Invoking the machinery of CP-chains yields a projection theorem for Bernoulli measures, which generalises in part earlier results by Hochman–Shmerkin and Ferguson–Jordan–Shmerkin. We also give an ergodic theoretic criterion for the dimension part of Falconer's distance set conjecture for general sets with positive length using CP-chains and hence verify it for various classes of fractals such as self-affine carpets of Bedford–McMullen, Lalley–Gatzouras and Barański class and all planar self-similar sets.

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1. Introduction and main results

Using ergodic theory to study problems in geometry is not new, however, there have recently been some major advances in the fields of fractal geometry and geometric measure theory made by studying the dynamics of the process of *magnifying* fractal sets and measures. In particular, the recent papers of Hochman [19] and Hochman–Shmerkin [20] have developed ideas of Furstenberg [16,17] and introduced a rich theory which has proven most useful in solving many long standing problems in geometry and analysis where scaling dynamics is present. See for example [21] for applications to equidistribution problems in metric number theory and [10,20] for applications to projections of fractal sets and measures.

The idea of using tangents has been used extensively in the past, for example in the use of tangent measures in [34] and in metric geometry in the study of tangent metric spaces. Often it turns out that tangents enjoy more regularity than the original object of study. When the object of study has a conformal structure, for example in the case of self-similar measures satisfying the open set condition, one expects the tangents to be essentially the same object. However, in the presence of non-conformality, in the limit we often obtain a completely different but sometimes more regular geometry.

In order to take advantage of the new more regular limit geometry arising in the non-conformal setting, we have to find a way to transfer information back to the original object. The scaling dynamics of the blow-ups can be modelled using a *CP-chain*, which is a Markov process that records both the point where we zoom-in and the *scenery* which we then see. If the CP-chain enjoys suitable irreducibility properties, the main results of [20] yield that it is possible to transfer strong geometric information about the *projections* of the limit structures back to the original geometry of interest. In this paper, we find out that a class of non-conformal measures will have scaling statistics that satisfy the assumptions required by Hochman and Shmerkin, which allows us to obtain new geometric results about them.

Our main applications will concern the Hausdorff dimensions of various sets and measures related to classical problems in geometric measure theory. Throughout the paper we will write dim for the Hausdorff dimension of a set and for the *(lower) Hausdorff dimension* of a measure, which is defined as

$$\dim \mu = \inf\{\dim E : \mu(E) > 0\}.$$

We will also write $\mathcal{H}^s$ for the s-dimensional Hausdorff measure. For a review of these notions see [28].

1.1. *Scenery of* $\times m$ *and* $(\times m, \times n)$ *invariants*

Important examples of dynamically invariant sets and measures can be found by studying the expanding maps $T_m : \mathbb{T} \to \mathbb{T}$, $m \in \mathbb{N}$, of the unit circle $\mathbb{T}$, where

$$T_m(x) = mx \bmod 1.$$