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Difference Galois theory of linear differential equations ☆

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ABSTRACT

We develop a Galois theory for linear differential equations equipped with the action of an endomorphism. This theory is aimed at studying the difference algebraic relations among the solutions of a linear differential equation. The Galois groups here are linear difference algebraic groups, i.e., matrix groups defined by algebraic difference equations.

0. Introduction

Linear differential equations with coefficients in a differential field (K, δ) and their behavior under the action of an endomorphism σ of K are a frequent object of study. Let us start with some classical examples. For instance, one can consider the field $K = \mathbb{C}(\alpha, x)$ of rational functions in the variables α, x and equip K with the derivation $\delta = \frac{d}{dx}$

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and the endomorphism $\sigma : f(\alpha, x) \mapsto f(\alpha+1, x)$. The Bessel function $J_\alpha(x)$, which solves Bessel's differential equation

$$x^2\delta^2(y) + x\delta(y) + \left(x^2 - \alpha^2\right)y = 0$$

satisfies the linear difference equation

$$xJ_{\alpha+2}(x) - 2(\alpha+1)J_{\alpha+1}(x) + xJ_\alpha(x) = 0.$$

Contiguity relations for hypergeometric series provide a large class of examples in a similar spirit. (See for instance [51, Chapter XIV].)

Another occasion, where a linear differential equation comes naturally equipped with the action of an endomorphism arises in the p-adic analysis of linear differential equations, when considering Frobenius lifts. For example, let p be a prime number and let us consider the field $\mathbb{C}_p$ with its norm $|\ |$, such that $|p| = p^{-1}$, and an element $\pi \in \mathbb{C}_p$ verifying $\pi^{p-1} = -p$. Following [19, Chapter II, §6] the series $\theta(x) \in \mathbb{C}_p[[x]]$, defined by $\theta(x) = \exp(\pi(x^p - x))$, has a radius of convergence bigger than 1. Therefore it belongs to the field $\mathcal{E}^\dagger_{\mathbb{C}_p}$, consisting of all series $\sum_{n\in\mathbb{Z}} a_n x^n$ with $a_n \in \mathbb{C}_p$ such that

- $\exists\varepsilon > 0$ such that $\forall\rho \in]1, 1+\varepsilon[$ we have $\lim_{n\to\pm\infty} |a_n|\rho^n = 0$ and
- $\sup_n |a_n|$ is bounded.

One can endow $\mathcal{E}^\dagger_{\mathbb{C}_p}$ with an endomorphism $\sigma\colon \sum_{n\in\mathbb{Z}} a_n x^n \mapsto \sum_{n\in\mathbb{Z}} a_n x^{pn}$. (For the sake of simplicity we assume here that σ is $\mathbb{C}_p$-linear.) The solution $\exp(\pi x)$ of the equation $\delta(y) = \pi y$, where $\delta = \frac{d}{dx}$, does not belong to $\mathcal{E}^\dagger_{\mathbb{C}_p}$, since it has radius of convergence 1. Moreover, $\exp(\pi x)$ is a solution of an order one linear difference equation with coefficients in $\mathcal{E}^\dagger_{\mathbb{C}_p}$, namely:

$$\sigma(y) = \theta(x)y.$$

So, here is another very classical situation in which one considers solutions of a linear differential equation and finds difference algebraic relations among them. (Coincidentally, in the above two examples the difference algebraic relations are linear.)

Understanding the relations among solutions of an equation is a question which is very much in the spirit of Galois theory. In this article we introduce a Galois theory which is able to handle linear differential equations in situations like the ones described above. More precisely, we develop a Galois theory which deals with the difference algebraic relations among solutions of linear differential equations. The Galois groups here are linear difference algebraic groups, i.e., matrix groups defined by algebraic difference equations.

Galois theories for various types of equations have become available over the years. The classical Galois theory of linear differential (or difference) equations, also known as